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SUMMARY TECHNICAL REPORT
OF THE
NATIONAL DEFENSE RESEARCH COMMITTEE

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SUMMARY TECHNICAL REPORT OF DIVISION 7, NDRC

VOLUME 1

GUNFIRE CONTROL

OFFICE OF SCIENTIFIC RESEARCH AND DEVELOPMENT
VANNEVAR BUSH, DIRECTOR

NATIONAL DEFENSE RESEARCH COMMITTEE
JAMES B. CONANT, CHAIRMAN

DIVISION 7
H. L. HAZEN, CHIEF

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WASHINGTON, D. C., 1946

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NOTES ON THE ORGANIZATION OF NDRC

The duties of the National Defense Research Committee were (1) to recommend to the Director of OSRD suitable projects and research programs on the instrumentalities of warfare, together with contract facilities for carrying out these projects and programs, and (2) to administer the technical and scientific work of the contracts. More specifically, NDRC functioned by initiating research projects on requests from the Army or the Navy, or on requests from an allied government transmitted through the Liaison Office of OSRD, or on its own considered initiative as a result of the experience of its members. Proposals prepared by the Division, Panel, or Committee for research contracts for performance of the work involved in such projects were first reviewed by NDRC, and if approved, recommended to the Director of OSRD. Upon approval of a proposal by the Director, a contract permitting maximum flexibility of scientific effort was arranged. The business aspects of the contract, including such matters as materials, clearances, vouchers, patents, priorities, legal matters, and administration of patent matters were handled by the Executive Secretary of OSRD.

Originally NDRC administered its work through five divisions, each headed by one of the NDRC members. These were:

Division A — Armor and Ordnance
Division B — Bombs, Fuels, Gases, & Chemical Problems
Division C — Communication and Transportation
Division D — Detection, Controls, and Instruments
Division E — Patents and Inventions

In a reorganization in the fall of 1942, twenty-three administrative divisions, panels, or committees were created, each with a chief selected on the basis of his outstanding work in the particular field. The NDRC members then became a reviewing and advisory group to the Director of OSRD. The final organization was as follows:

Division 1 — Ballistic Research
Division 2 — Effects of Impact and Explosion
Division 3 — Rocket Ordnance
Division 4 — Ordnance Accessories
Division 5 — New Missiles
Division 6 — Sub-Surface Warfare
Division 7 — Fire Control
Division 8 — Explosives
Division 9 — Chemistry
Division 10 — Absorbents and Aerosols
Division 11 — Chemical Engineering
Division 12 — Transportation
Division 13 — Electrical Communication
Division 14 — Radar
Division 15 — Radio Coordination
Division 16 — Optics and Camouflage
Division 17 — Physics
Division 18 — War Metallurgy
Division 19 — Miscellaneous
Applied Mathematics Panel
Applied Psychology Panel
Committee on Propagation
Tropical Deterioration Administrative Committee

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NDRC FOREWORD

As EVENTS of the years preceding 1940 revealed more and more clearly the seriousness of the world situation, many scientists in this country came to realize the need of organizing scientific research for service in a national emergency. Recommendations which they made to the White House were given careful and sympathetic attention, and as a result the National Defense Research Committee [NDRC] was formed by Executive Order of the President in the summer of 1940. The members of NDRC, appointed by the President, were instructed to supplement the work of the Army and the Navy in the development of the instrumentalities of war. A year later, upon the establishment of the Office of Scientific Research and Development [OSRD], NDRC became one of its units.

The Summary Technical Report of NDRC is a conscientious effort on the part of NDRC to summarize and evaluate its work and to present it in a useful and permanent form. It comprises some seventy volumes broken into groups corresponding to the NDRC Divisions, Panels, and Committees.

The Summary Technical Report of each Division, Panel, or Committee is an integral survey of the work of that group. The first volume of each group's report contains a summary of the report, stating the problems presented and the philosophy of attacking them, and summarizing the results of the research, development, and training activities undertaken. Some volumes may be "state of the art" treatises covering subjects to which various research groups have contributed information. Others may contain descriptions of devices developed in the laboratories. A master index of all these divisional, panel, and committee reports which together constitute the Summary Technical Report of NDRC is contained in a separate volume, which also includes the index of a microfilm record of pertinent technical laboratory reports and reference material.

Some of the NDRC-sponsored researches which had been declassified by the end of 1945 were of sufficient popular interest that it was found desirable to report them in the form of monographs, such as the series on radar by Division 14 and the monograph on sampling inspection by the Applied Mathematics Panel. Since the material treated in them is not duplicated in the Summary Technical Report of NDRC, the monographs are an important part of the story of these aspects of NDRC research.

In contrast to the information on radar, which is of widespread interest and much of which is released to the public, the research on subsurface warfare is largely classified and is of general interest to a more restricted group. As a consequence, the report of Division 6 is found almost entirely in its Summary Technical Report, which runs to over twenty volumes. The extent of the work of a Division cannot therefore be judged solely by the number of volumes devoted to it in the Summary Technical Report of NDRC; account must be taken of the monographs and available reports published elsewhere.

The Fire Control Division, initially Section D2 under the leadership of Warren Weaver and later Division 7 under Harold L. Hazen, made significant contributions to an already highly developed art. It marked the entrance of the civilian scientist into what had hitherto been regarded as a military specialty.

It was one of the tasks of the Division to explore and solve the intricate problems of control of fire against the modern military aircraft. Gunnery against high-speed aircraft involves fire control in three dimensions. The need for lightning action and superlatively accurate results makes mere human skills hopelessly inadequate. The Division's answer was the development of the electronic M9 director which, controlling the fire of the Army's heavy AA guns, proved its worth in the defense of the Anzio Beachhead and in the protection of London and Antwerp against the Nazi V-weapons. In addition to producing mechanisms such as the M9, the Division made less tangible but equally significant contributions through the application of research methods which had a profound, even revolutionary, influence on fire control theory and practice.

The results of the work of Division 7, formerly Section D2, are told in its Summary Technical Report, which has been prepared at the direction of the Division Chief and has been authorized by him for publication. It is a record of creativeness and devotion on the part of men to whom their country will always be grateful.

VANNEVAR BUSH, Director
Office of Scientific Research and Development

J. B. CONANT, Chairman
National Defense Research Committee

FOREWORD

As the termination of the war approached, the members of Division 7, most of whom had left active professional work to engage in full-time work for Division 7, were subjected to strong pressure to pick up their normal peace-time activities. Their obligations were so heavy that the task of getting out a Summary Technical Report was formidable for a number of the areas of Division 7 activities. That this first volume has actually been produced is due in large measure to the continued effort of Mr. L. M. McKenzie, the Division's editor for its Summary Technical Report and the person who has done a major part of the work of preparing Volume 1. His competence as Technical Aide in the Washington office of the Division inevitably brought him in close touch with much of the Division's activity. He therefore had an excellent background for taking over the responsibility of assembling this volume. For doing this task under difficult circumstances and doing it well, the members of Division 7, and particularly its Chief, are greatly indebted to Mr. McKenzie.

H. L. Hazen
Chief, Division 7

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PREFACE

THE SUMMARY TECHNICAL REPORT of Division 7, NDRC, comprises three volumes. They differ considerably from one another in method of presentation as well as in subject matter. The present volume briefly summarizes the work of Sections 7.1, 7.3, 7.5, and 7.6 of Division 7, and also lists contract numbers, service control numbers, personnel, and other factual data relating to the Division. Volume 2, dealing with optical range finders, reports in detail the work of Section 7.4. Volume 3 gives an extensive account of the activities of Section 7.2 on airborne fire control systems.

Throughout Part I of the present volume the reader is merely introduced to the researches and developments sponsored by a Section. He is expected to consult the references for details about a particular subject. This should not be objectionable, for the Division has throughout its life issued *Reports to the Services* summarizing in some detail the work in a particular field or upon a particular device. These have been widely distributed and are, with various other reports, included in the microfilmed record.

Part II of this volume, written by Dr. R. B. Blackman, Dr. H. W. Bode, and Dr. C. E. Shannon of the Bell Telephone Laboratories, departs from this pattern. It is a detailed technical treatise on smoothing of data and represents material which, if written during the war, would have been issued as a report to the Services. It is included here since it highlights a particular contribution to the fire control art which was of especial importance in the activities of the Division.

In keeping with the general character of the present volume, Dr. Harold L. Hazen, Chief of Division 7, has written the opening chapter. This bears upon all three volumes by orienting the reader with respect to the activities of the Fire Control Division. With the future worker in mind, he highlights those aspects of the Division 7 program which are particularly significant.

Finally, a word with regard to the inevitable errors and inconsistencies which may be present in the remainder of the text. Although compiled principally from the writings of other technical aides and members of the Division, certain interpretations and explanations may not faithfully reflect original meanings, and the writer assumes the responsibility for any distortion. In particular, Duncan J. Stewart and George R. Stibitz contributed to Chapter 2, E. J. Poitras to Chapters 3 and 4, Warren Weaver and George R. Stibitz to Chapter 5, and Ivan A. Getting to Chapter 6.

L. M. McKENZIE
Technical Aide, Division 7

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PART I

GUNFIRE CONTROL

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Chapter 1

FIRE-CONTROL ACTIVITIES OF DIVISION 7, NDRC

By *H. L. Hazen*

UNDER THE URGENCY of war, Division 7 of NDRC, in common with other development agencies, stressed useful results rather than integrated summary records. Reports there were in great numbers, many of them excellent and valuable, but the broad objectives and overall plan for the division's work were carried largely in men's minds. Each of the division members knew which projects were improvisations against time, which were aimed at fundamental gains, and which were speculative ventures with promising but unproven ideas. If the enduring values contained in the many hundreds of man-years' work of able scientists and engineers invested in the division's program are to be of continuing use, they must be reasonably accessible to the future worker. To provide this accessibility, an overall view of the program is needed by means of which the future worker can see from the division's point of view the objectives, nature, and significance of the various projects, and know to what sources to go to find the pertinent results. It is the purpose of this Summary Technical Report to supply this overall view and to point out these sources.

To accomplish this purpose best within the limitations of manpower available for preparing this report, it has been necessary to permit a treatment that is nonuniform in style and approach. In some areas the work has been previously reported very fully in the form of Reports to the Services so that a brief statement for orientation and introduction to the bibliography suffices. In other areas the individual reports have treated specialized details so that here a comprehensive logical exposition seems necessary. Thus this summary is designed to complement previous reports in such a way that the two taken together form a consistent whole.

This chapter serves as an introduction to the three volumes of the Division 7 Summary Technical Report. It is an attempt to high light and appraise in such a way that the reader may know what the author in retrospect regards as the more important subjects for future attention. First, however, certain observations concerning the nature of fire control and its development prior to World War II will perhaps serve to establish common ground even though they may be a statement of the obvious.

1.1 FIRE CONTROL

1.1.1 Scope of Fire Control

Fire control is here understood to include means of directing missiles of all sorts in such a way as to cause them to strike desired targets. Thus it is associated with all forms of guns, with rockets, with bombs dropped from airplanes whether they be free-falling or guided, with torpedoes, with depth charges dropped from ships or airplanes, to mention some of the more important categories. While the work of the division was associated with all of these areas, the intensity of effort varied widely and in a number of areas other divisions involved in missile development did the major part of the fire-control work for their missiles.

Fire control has reached its highest stage of evolution thus far in the heavy antiaircraft artillery field. There are excellent reasons why this is so. First, antiaircraft action is so rapid, the problem is so complicated, and the results are required to be so accurate that simple human skill is utterly useless. Very high quality instrumentation alone has any chance of success. In contrast, field artillery, for example, achieves high precision but its targets are essentially stationary and the time for working up fundamental data is relatively great. Furthermore, the field artillery problem is largely two-dimensional in character. The Navy main-battery surface-fire problem may involve moving targets and is complicated by a moving gun platform, but here the problem is still two-dimensional and the time scale is relatively expanded in comparison with that for antiaircraft fire.

Control of various forms of fire from aircraft is an equally difficult problem in its technical elements but the stringent weight and space limitations on airborne equipment preclude the elaborateness and refinement permissible in the heavy surface and antiaircraft fire control.

1.1.2 Dynamical Unity of a Fire-Control System

Fire control involves very complex physical apparatus, among the most complex that has been devised. Because of this, as well as of the security aspect mentioned subsequently, the theory of dynamical behavior was in a very primitive state prior to World War II. This has meant that there frequently has been difficulty in distinguishing malfunctioning due to mechanical imperfection or design from that due to violation of dynamical laws. One aspect of this, the essential dynamical unity of a *complete* fire-control system is so important as to justify the following remarks.

Functionally, fire-control apparatus may be roughly separated into three components. The first is the data-gathering elements which provide information concerning the target position and motion, own-ship's orientation, position, and velocity, and secondarily wind, air density, and other correction data. The second component performs the computation functions. It determines from all the data furnished by the first component how the missile shall be launched in order to arrive at some given point in space at the same instant that the target is at that point. The third component, not always thought of as strictly fire control, includes the data-transmission and power elements by which the data calculated by the computer are caused to control the gun, the automatic pilot or other means of launching the missile.

The separation of the three components, for purposes of analysis and understanding, is justifiable on much the same basis that one separates in a living organism for purposes of study the circulatory, respiratory, and nutritive functions. One must always remember, however, that a fire-control system, just as a living organism, is more than the sum of component parts. It is an integrated whole with interrelated functioning of all its parts and one is safe in considering the parts separately only if one always keeps in mind their relation to the whole. This is a fundamental fact of utmost importance.

1.1.3 Future Use of Fire Control

One may, and in fact should, ask wherein the fire control of World War II is pertinent to the methods of warfare we shall have to use in the future if mankind continues so primitive as to find fighting the only way of settling disputes. The future warfare appears to place great emphasis on long-range missiles of great destructive power guided largely automatically to their targets — perhaps automatically seeking them. This war's fire control is pertinent because even if the specific fire-control apparatus used in connection with this war's ordnance were not applicable, the fundamental principles of fire control for guided missiles are precisely the same as those underlying the fire control of World War II. Component mechanisms and techniques are essentially the same. Furthermore, the concept of dynamical unity is even more comprehensive in its scope in the guided missile problem, for here it must include guiding and homing functions, in addition to the three component functions of conventional fire control. Therefore, the fire control of the future will be merely an extension and outgrowth of fundamental principles and techniques of today. For this reason it appears worth while to attempt to make the results of the substantial efforts in fire control during World War II accessible to future workers. A second, but not a negligible, reason why this war's fire control is pertinent, is that history seems to indicate that old and seemingly primitive techniques often retain their value in war for a surprisingly long time after newer methods have made their appearance. That is, we are far from sure that essentially conventional antiaircraft artillery, conventional bombing by airplanes, and conventional plane-to-plane gunfire are obsoleted by newer developments.

1.2 PREWAR STATUS OF FIRE CONTROL

The role that Section D-2 and Division 7 played in World War II can perhaps be better understood in terms of the prewar status of fire control. Viewed as a technical field, there are two outstanding characteristics of fire control. First, it is a highly technical subject requiring an extraordinarily high level of competence in the research, development, and production phases on the part of professional scientists, engineers, and craftsmen. Second, it is by nature one of the more highly classified and isolated areas of military endeavor. The latter is naturally so because of the very great importance of fire control in securing domination of an enemy and because the almost inspired type of creativeness required makes the preservation of security highly important on the part of those who have achieved high competence.

Near the beginning of World War II, the United States was favored by having a few groups who under a tight veil of secrecy had shown ingenuity and skill amounting practically to genius in the development and production of fire-control gear. Their mechanical design and craftsmanship show a courage, a persistence in the face of formidable difficulties, a mastery of complication, and a refinement in execution that command the highest respect. However, the isolation inevitably bred by security necessarily cut off these groups not only from association with scientists and engineers at large, but even from each other. These are naturally not the conditions that yield the best that science and technology can produce. However, it did produce surprisingly good results in this country even with the severe restrictions that were imposed.

When Section D-2 entered the scene, it found this highly developed art characterized by skill and craftsmanship of superlative grade, particularly in mechanical design and construction. It was an accomplishment in which any group working under these conditions could justifiably take great pride. Anyone entering the field fresh even with an extraordinarily strong scientific background would necessarily find himself faced with months of hard study merely to catch up with what had been done. The field of fire control had been worked over intensively and with great ingenuity within the scope of techniques possessed by the small quota of individuals who were given access to and permitted to work in this field. The thing that new groups broadly grounded in science might be expected to bring to such a field are the benefits of a new and fresh point of view and of a range and breadth of experience over a variety of fields that could see relations between fire control and many varied fields of endeavor that, superficially viewed, are unrelated to it.

A good example of the benefit of a fresh point of view is the development that led to the antiaircraft director M9 for the Army's heavy antiaircraft guns. This development represented two significant departures from previous work. The first is the bringing to bear of communications techniques on the problem of smoothing and prediction. The second is the development of electrical computing techniques to replace the previous mechanical techniques.

Superficially there is no apparent relation between fire control and electrical communications. More fundamentally, however, both are concerned with the separation of useful information or data from the unwanted but unavoidable data in the form of "noise" or rough tracking. In fact, ultimate performance of equipment in both fields is limited fundamentally by the extent to which these two, the wanted information and the unwanted or spurious information, can be separated. The adaptation of the methods already highly developed for this purpose in the communications field into a form useful in the fire-control field constitutes one of the important contributions of World War II to fire control. For this reason, the comprehensive and fundamental monograph on data smoothing and prediction[a] is included as Part II of Volume 1.

[a] By R. B. Blackman, H. W. Bode, and C. E. Shannon of the Bell Telephone Laboratories.

There are numerous other examples of the benefits obtained by cross-fertilization between fields. It is believed that the personnel of Section D-2 and Division 7, representing as they did rather diverse backgrounds coupled in each case with a broad scientific training, were able to bring about substantial contributions in many areas of fire control.

1.3 DIVISION 7 CONTRIBUTIONS

1.3.1 Developments Put in Service

One may ask what specific contributions Division 7 made to the winning of World War II. There are a few concrete items and, in addition, a large number of intangibles whose contribution it is impossible to assess with accuracy.

In the tangible category the antiaircraft director M9 which formed so successful a partner with the SCR-584 radar and the proximity fuse undoubtedly played an important role on a number of occasions among which may be mentioned the antiaircraft defense at the Anzio Beachhead and the V-1 defense of England and Antwerp. Section D-2 played a major role in the development of this director by the Bell Telephone Laboratories.

Two other Division 7 developments that reached the active theaters in substantial quantities are the oil-gear power drive M3B1 and the M7 sighting system both for the 37-mm and 40-mm guns. The first of these was developed at the Massachusetts Institute of Technology and produced by Westinghouse. The second was developed by Pitney-Bowes. The oil-gear drive was apparently very satisfactory, since it went into the field in large numbers with practically no resulting complaints. The M7 sighting system (the course-and-speed sight often known as the Weissight) produced varying responses as did the other competitive fire-control systems used with the Bofers gun. None of these was ideal.

1.3.2 Fundamental Performance Data

A second and somewhat less tangible contribution of Division 7, though probably no less real, was the influence exerted by increasing emphasis on the vital importance of meaningful quantitative data on fire-control system performance. Prior to World War II no equipment existed for testing fire-control systems or even important components under dynamical reproducible conditions. This emphasis was reinforced and implemented by a series of developments of testing machines. Conspicuous among these are the Barber-Colman and the Bell Telephone Laboratories dynamic testers for antiaircraft fire control; the Texas tester for plane-to-plane fire control; and the Patuxent River plane-to-plane fire-control testing establishment set up for the Navy. These and various less imposing similar developments, together with the division's perpetual insistence upon significant appraisal, undoubtedly exerted major influence on fire-control thinking both in the United States and abroad.

1.3.3 Consulting Activity

A third major but quite intangible contribution is the influence exerted by essentially consulting activities with many branches of both Services by able men associated with the division. Scientific and technical counsel from division personnel were sought by many Service groups on innumerable occasions prior to the making of important decisions. The overall effect of such consultation of the war effort is impossible to appraise. It is believed to have been substantial.

1.4 SUGGESTIONS FOR FUTURE WORK

After these preliminary observations, we turn now, in the remainder of this chapter, to a rapid review of some of those aspects of the division's work to which it is believed the future worker can with profit give some attention. To avoid undue complexity and detail which might obscure the main points, only the more important items are mentioned.

1. Fundamental theory.
2. Testing and appraisal.
3. New principles and components.
4. Matching mechanisms to men.
5. Optical range finders.

For the reader who desires further information more detail is given in subsequent chapters of this Summary Technical Report. From this, in turn, the reader can go to the original reports.

1.4.1 Fundamental Theory

One of the more important studies in this field is that mentioned previously, namely the monograph on data smoothing and prediction (Part II of this volume). Another fundamental study of the prediction process[b] is summarized in Chapter 5 of this volume. These are probably the two most important efforts of the division in fire-control theory. This work has had and will continue to have a fundamental influence on subsequent thinking.

From a somewhat different point of view fire-control theory has been influenced substantially by statistical studies such as are reviewed in Chapter 5. The measure of the effectiveness of ordnance is the probability of damage to a target which depends on many factors including operating procedures and tactics as well as fire control. Studies showing how each factor affects the probability of damage to target are therefore of great importance as a guide to development showing where effort should be placed to secure the greatest gains. Much work was done in this field, including some by Section 7.5 of Division 7, as summarized in Chapter 5.

In the airborne fire-control field the substantial contributions to fundamental theory[c] are treated in a unified manner in Part I of Volume 3. This treatment embraces nearly all phases of airborne fire control including gunfire, bombing, rocketry, and aerial torpedoing, and represents much effort and analysis. Incidentally, the brief section on torpedo fire-control theory,[d] Part II of Volume 3, should not be overlooked as a possibility for capitalizing on the high-grade "present-range" information available by radar.

Under the subject of fundamental theory should be noted the several publications sponsored by Division 7 on the theory of servomechanisms. These have been widely circulated in response to many requests.

Other contributions to the fundamental theory of fire control are found scattered throughout the division's reports but the foregoing indicates the more comprehensive contributions.

1.4.2 Testing and Appraisal

As has been stated, the division considers that one of its major contributions is the strong and persistent emphasis on meaningful quantitative data as the only sound basis for appraisal of the technical performance of fire-control equipment. The two principal areas in which this feeling was influential are in the heavy antiaircraft problem and the problem of plane-to-plane fire control. Relatively late in World War II preliminary studies were made for adequate instrumentation in the automatic weapons gunnery field and in the bombing field. In the latter two, however, the adequate instrumentation appeared to require a longer term of development than was justified by the stage the war had reached. The bombing analyzer study was undertaken preliminarily by the Eastman Kodak Co. The automatic weapons accessory problem was given exploratory study by the Armour Research Foundation.

Returning to testing in the antiaircraft artillery field, Section D-2 early became impressed with both the exceeding costliness and the inadequacy of existing testing and appraisal methods. These were limited primarily to the quite unsatisfactory towed-target field tests and to component tests consisting largely of static measurements. The prediction process fundamentally requires elements whose dynamic performance in response to rapidly changing data is essentially different from its static performance; consequently the only significant tests are those that effectively simulate dynamic conditions. The mechanical dynamic tester developed by the Barber-Colman Co. and the two models of tape dynamic tester later developed by the Bell Telephone Laboratories marked very important steps in this process of obtaining adequate instrumentation

[b] By Norbert Wiener.
[c] By George A. Philbrick.
[d] By A. L. Ruiz.

for dynamic tests of fire-control equipment under reproducible conditions. Implementing these and the involved overall trial firing tests are a series of computing machines and aids, a data recorder, and other auxiliary instruments by means of which these processes are speeded up and their time and money costs reduced. Both the dynamic tester and these computing aids are considered to be important tools in future developments.

As pointed out above, however, the testing of fire control for automatic weapons is in a most unsatisfactory state. Serious attention should be given to it in the future.

In the field of the plane-to-plane fire control even more expensive and elaborate developments were required than for heavy antiaircraft, in order to provide even a good first approximation to satisfactory testing and assessment techniques. Two major developments of the division stand out as of continuing significance, namely, the Texas tester and the Patuxent River testing project.

1.4.3 New Principles and Components

Out of the numerous projects carried on by the division came many new ideas or modifications of old ideas which did not achieve practical embodiment by the end of World War II. A number of these ideas, however, are believed to be worth the attention of future workers in the field and are pointed out here for that purpose.

In the field of antiaircraft fire control (under Section 7.1 of Division 7) which is covered in Chapter 2 of the present volume, particular attention is called to the following projects. The report of the Bryant Chuck Grinder Co. contains a substantial number of novel suggestions for fire-control mechanisms, some quite unconventional, that deserve careful study. While the M9 director proved the value of electrical computation techniques in fire control, it by no means exhausted the possibilities of electrical computation; in fact it serves merely as an introduction to a field that justifies far more extensive exploitation. The experimental T15 director contains ideas worthy of merit as do directors involving electrical techniques produced experimentally by a number of groups under direct contracts with the Services and having no connection with NDRC. Certainly further attention in the heavy antiaircraft fire-control field should be given to other than M9 electrical techniques as well as to possible new mechanical computer techniques and, probably more important, various combinations of electrical and mechanical means of computation.

The prediction of future position of targets flying accelerated paths is one which has arisen in the past and will undoubtedly arise frequently in the future. On this subject the extensive series of investigations undertaken by the Bell Telephone Laboratories under Division 7 and Ordnance Department auspices are worthy of attention.

In the automatic weapons field, but having implications for heavier guns, are the T28 director developed by Eastman Kodak Co. and the course-invariant sight developed by the Baker Manufacturing Co. The T28 embodies a direct representation of three-dimensional vector quantities that seems inherently adapted to a problem that is fundamentally three-dimensional in character.

The Baker course-invariant sight also involves novel principles having certain inherent advantages that appear to justify further development.

In the naval field, the gunfire-control system Mark 56 is believed to represent a significant advance in the fire control of heavy Navy and antiaircraft guns. This set is going into limited production so that appraisal under sea conditions will be possible. It benefited from the major advances in fire-control thinking achieved during World War II.

Turning now to airborne fire control, there is an extended series of devices developed at the Franklin Institute and elsewhere for gunnery, bombing, torpedoing, and rocket firing that near the end of the war were forming into a single pilot's universal sighting system [PUSS]. These are covered in Volume 3, Chapter 10. This work may be of value for two reasons. First, the successful ideas should be continued, and second, there is an extensive series of negative results which can be reviewed with benefit by anyone seriously interested in this field.

In the plane-to-plane gunnery field, the work of the Franklin Institute applied primarily to fixed gun installations. In the field of turret gunnery, the developments at the General Electric Co. on airborne computers contain numerous ideas of interest in the future development of airborne central-station computers.

In calling attention to new principles and components that may be of interest in future work, some emphasis can well be placed on the extensive exploration and partial exploitation of pneumatic techniques by Section 7.3 of Division 7, NDRC. The low-level bombsights, Mark 23 and Mark 25, represent the fire-control uses of pneumatic methods for computation and energy amplification. In the field of lead-computing sights for either surface or possibly aerial use, the pneumatic methods partially investigated under the Eastman Kodak Co. project are worth further attention. It is believed that in various computation and servo applications, particularly where a fast response and light weight are at a premium, pneumatic methods are worth very careful examination and substantial further development. These matters are discussed in Chapter 4.

Section 7.3 also sponsored a considerable amount of work on servomechanisms other than the pneumatic type. These are discussed in Chapter 3. The gun drives for the Army 37- and 40-mm guns, for example, achieved notable success.

1.4.4 Matching Mechanisms to Men

Growing out of the major activity on the subject of optical range finders was a substantial amount of work on the combined performance of operators and machines. In particular, many investigations were aimed at determining what properties or parameters of machines, such as tracking devices, for example, enabled the man to produce most precise results with the machine. Of these studies the extensive work on tracking done by the Foxboro Company (under Section 7.4, reported in Volume 2) and the Franklin Institute (under Section 7.2, reported in Volume 3) deserve particular attention from persons interested in the manual tracking process. These studies have produced results in such form as to be in many cases immediately applicable for designers of manually controlled fire-control equipment. The techniques employed are also suggestive for those who wish to pursue this subject farther.

1.4.5 Optical Range Finders

While by the end of World War II the spectacular success of radar methods of determining target position had demonstrated their accuracy, reliability, and resistance to countermeasures (when intelligently designed and used), it was not until nearly then that these qualities of radar had sufficiently demonstrated themselves to obviate intensive work on optical range finders. Range finder work was started very early by the division because preliminary examination of the fire-control problem had clearly indicated that the largest instrumental contributor to errors in antiaircraft firing was the optical range finder. While much of this work will be of interest only if optical range finders should for some unforeseen reason again assume major importance in warfare, there are certain results that have future interest. One of these is the Harvard University studies on the fundamental mechanisms of visual distance determination. The results there obtained have a profound significance in relation to any future range finder development because they extend our knowledge of the fundamental distance perception process into entirely new territory.

The benefits of another major cooperative enterprise among the Army, the Navy, the three major range finder manufacturers, and Division 7 have already been passed on to the Services who took over the Division 7 contracts with the three range finder manufacturers. This so-called "super range finder" project aimed to embody all the results previously obtained in a series of designs for superior performance. This work had not felt the impact of the Harvard work, which indeed calls for rather revolutionary changes in optical range finders if the highest possible precision is to be obtained.

The work associated with optical range finders under Division 7 is tremendous in bulk, but Volume 2 of this report[e] should enable the worker interested in the subject to gain access to pertinent material in this field with relative ease.

1.5 CONCLUDING REMARKS

In conclusion a few words about the method of operation of Division 7 may be of interest. Almost complete autonomy was granted to the various divisions of NDRC in their method of operation, with the result that each division largely developed its own pattern for work. Division 7 consisted of a group of professional scientific people who, with few exceptions, devoted full time to the scientific and technical direction of the Division 7 activity. In addition they carried the necessary administrative responsibility as far as policy matters were concerned. This group, assisted by technical aides and frequently by contractors' personnel, collected information, originated ideas, formulated programs, and supervised projects in some detail. The project supervision was mainly exercised by mutual discussion of the problem with contractors' personnel, with a joint mapping of the future course. In other words, the division personnel served in an advisory and co-worker status rather than as a giver of orders. The latter was necessary only as a legality to formalize conclusions previously arrived at by mutual discussion and understanding. It can be said with entire honesty that the division-contractor relations under this arrangement were, with few if any exceptions, characterized by complete cordiality, frankness, and complete absence of personal conflict or serious friction.

Division meetings in general were held once a month. They were usually for one day although not infrequently a two-day meeting was held at some center of interest, with one day devoted to business, and the second to reviewing the work of some project or contractor. Characteristically, the meetings were very free, uninhibited discussions of projects, technical problems and policies, carried on between men who were good personal friends but who were at the same time strong individualists with well-considered convictions. Most conclusions were reached only after thorough discussion, which occasional visitors were sure involved personal rancor until they learned that Division 7 business was done in this way without acrimony. These sessions were very stimulating.

The various sections within the division each had followed their own individual patterns of activity. Several, namely Sections 7.1, 7.3, and 7.4, functioned informally. Their work was done at frequent but irregular meetings of two or three persons. Section 7.2, on the other hand, operated in much the same way as the division itself. It ordinarily held its meetings the day before the division meeting with the result that its recommendations to the division were usually clearly formulated and in a mature state when they came up for division discussion and action. Section 7.6, which was only beginning to acquire vigor when the first false OSRD demobilization wave struck, also had formal meetings at which technical matters were discussed and programs formulated. Section 7.5 was an administrative convenience without expectation of regular section activity.

All in all the accomplishments of Division 7 are believed to be substantial. They are not so spectacular or so revolutionary as those of some of the other divisions. This may be due to the personnel or their method of operation. It is believed that at least one other factor had a rather prominent influence, however, and that is the relative maturity of the fire-control field as compared with some other fields, such as microwave radar, where almost completely new techniques were created.

In any case it is the judgment of the author of this chapter, who joined the division after its pattern had in many ways been well established, that whatever the results the division achieved, its activity was characterized by wholehearted immersion in and absorption by the work to be done, by no little imagination,

[e] By S. W. Fernberger.

by a high order of technical intelligence, and by at least a reasonable degree of success as judged by its overall record. As pointed out earlier in this chapter, its greatest influence and effectiveness was probably in its consulting services rather than in the apparatus developed.

The writer of this chapter is permitted to make the foregoing statements because he joined the division rather late and therefore can qualify with respect to most of the division's activity rather as an observer than as one who took a primary part in the solution of technical problems.

Chapter 2

LAND-BASED FIRE CONTROL

2.1 GENERAL ORGANIZATION OF ACTIVITIES OF SECTION 7.1

2.1.1 Evolution of Section 7.1 Responsibilities

REFERENCE to a pre-Pearl Harbor history of Section D-2 of the National Defense Research Committee [NDRC] shows the Fire-Control Committee as initially committed to investigation and development of land-based antiaircraft fire-control methods and means to the exclusion of airborne and seaborne fire-control devices.

Indeed, during the early days of Section D-2, studies were made of the fire-control field generally; but it was clear that the Army Ground Forces offered the greater opportunity for service and accomplishment. This conclusion was derived partially from a technical appraisal of the situation, but even more from the fact that the Army Ground Forces were less satisfied than the Navy or the Army Air Forces with the quality of its own equipment and more disposed to welcome cooperative efforts of Section D-2 in carrying out new developments. With the passing of some months, Section D-2 (later Division 7) undertook heavy programs in contribution to airborne and seaborne fire control, and also continued to expand its efforts in the field of land-based fire control. The termination of hostilities found Division 7 still deeply committed to this work under the auspices of Section 7.1.

The first program of Section D-2 contained two broad classes of work: (1) a study of fundamental problems, and (2) a theoretical and experimental attack on the specific development of a more effective fire-control system for heavy antiaircraft weapons. A completely orderly and systematic execution of the section program should have proceeded by first studying fundamental problems and following this by developing the necessary instruments. But the exigencies of the situation required the less ideal and more expensive approach of carrying out both aspects of the program simultaneously and in more or less opportunistic fashion, and the section undertook an immediate program of antiaircraft director development.

These early studies quickly required Section D-2 to broaden its program, and two further classes of work were undertaken. The first of these sprung from a lack of adequate instruments and procedures for the measurements, analyses, and assessments of the performance of fire-control instruments and components. The second was an extension of effort into the research and development of fire-control means for automatic weapons.

With the reorganization of Section D-2 as Division 7, Section 7.1 was made responsible for these classes of Section D-2 projects and two additional categories were to appear: (1) fire-control instruments for other weapons of a miscellaneous character, but pertaining to the land-based fire-control problem, and (2) certain short-base range finders which were begun under Section D-2 and Division 7 auspices, but which pertained directly to the land-based fire-control problem.

2.1.2 Classification of Projects

For the purposes of a summary technical report of Section 7.1 activities, the projects which were the responsibilities of that section will be treated in this chapter under the following six categories:

1. Major caliber antiaircraft weapons and gun directors.
2. Fundamental studies.
3. Automatic weapons.
4. Miscellaneous weapons.
5. Short-base range finders.
6. A testing program.

As indicated already, categories 1 and 2 are phases of a single problem and, although segregated, they should be considered as mutually supporting. This report attempts to summarize the nature of the problems involved under the various contracts and to describe the progress made. For details of these developments the reader will be referred to appropriate documents.

2.2 MAJOR CALIBER ANTIAIRCRAFT WEAPONS (GUN DIRECTORS)

2.2.1 Electrical Gun Director M9 (Project 2)

Perhaps one of the most outstanding projects of Section D-2, if not the most outstanding, was the development of the electrical gun director M9 by the Bell Telephone Laboratories. (Project 2, contract NDCrc-127.)

This project was undertaken primarily in the hope that it would be possible to produce an electrical design which would present two advantages over the standard mechanical design (the Sperry M7 director) : first, that it would be readily procured in large numbers, precision electrical equipment as a general rule being more easily fabricated than precision mechanical equipment, and second, that a director of improved performance would result. The second objective turned out to be of great importance, although at the time the project was undertaken, the limitations of and difficulties with the standard Army mechanical director M7 were not appreciated, particularly for use with radar. As it turned out, the first objective was not accomplished, for the new director was more expensive than the old.

In the model designated T10,[1] and in the resulting standard director M9, all conversions to rectangular coordinates, prediction, ballistic computations, and reconversions to polar coordinates are electrical. Present azimuth and elevation are normally obtained by manual operation (but with aided tracking) of a tracking head on which the operators ride (Figure 1). However, the necessary selsyns are provided so that, through manual matching of pointers, the tracking head may be driven from remotely received values of present azimuth and elevation. Movement of the tracking head operates potentiometers to insert voltages which are a function of the azimuth and elevation as inputs to the computer. Either slant range or height can be received from an M1

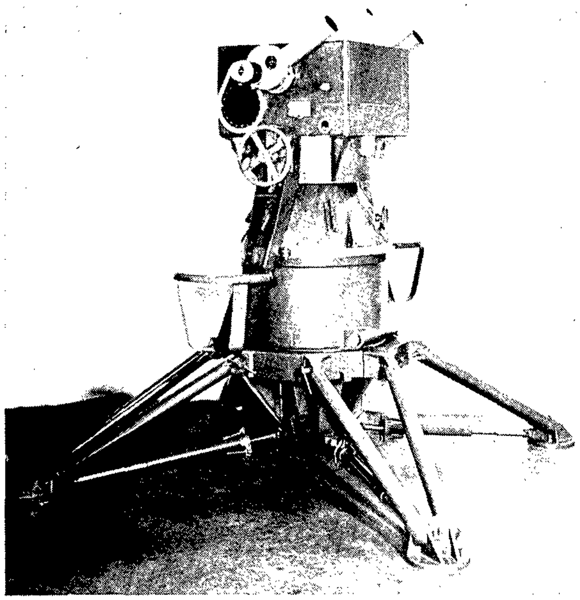


FIGURE 1. T10 director; the tracker.

height finder in a separate unit, and used with a manual follow-up to operate another potentiometer to supply a voltage as the range input to the computer. Radar range is also accepted directly from a potentiometer mounted on the range unit.

The prediction is theoretically exact and the sensitivity of individual amplifiers and potentiometers was held to approximately one-tenth of one per cent of full scale. All the computations, including many of those for the ballistics, depend upon the use of rather large, highly accurate, nonlinear potentiometers.

The ballistic computer (Figure 2) is very elaborate, being adjustable for wind, drift, parallax, muzzle velocity, and dead time. Large numbers of amplifiers and potentiometers are used in these computations in addition to those involved in the prediction. The outputs are converted to mechanical motions which drive the usual selsyn transmitters.

T10 was completed in November 1941, and a series of tests was carried on at the Antiaircraft Artillery Board at Fort Monroe, Virginia, immediately afterwards.[2] Although the results were not significantly better than those obtained from the standard Sperry director,

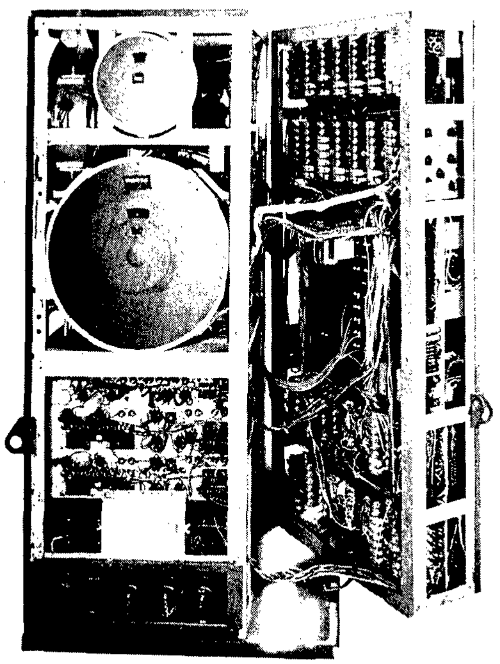


FIGURE 2. T10 director; view of computer showing amplifier frame detached for maintenance work.

without wind, drift, muzzle velocity, and other nonstandard conditions, analysis of the results indicated that the errors could be greatly reduced, perhaps to one-half their then existing value, by introducing improved smoothing circuits, the theory for which was developed under Projects 6 and 11 to be reviewed below.

Despite the unfinished nature of the experiment and the rather poor results, standardization was recommended, but at the same time an extension of the NDRC contract was provided to incorporate the new smoothing circuit and other changes in T10. This combination was tested and the expected improvement was realized. The modified director was also tested with radar XT-1 (later designated SCR-584) and the combination found to give very satisfactory results.

The M9 director embodied the improvements made in T10 and many others and, with SCR-584, gave an outstanding performance during World War II.

2.2.2 Gun Director T15 (Project 30)

During the development of the first electrical director under Project 2, it became apparent that there were many alternative electrical methods for solving the different problems which arose. The exploration of the alternative methods suggested a radically different approach than that used in the M9 electrical director and the development of a director,

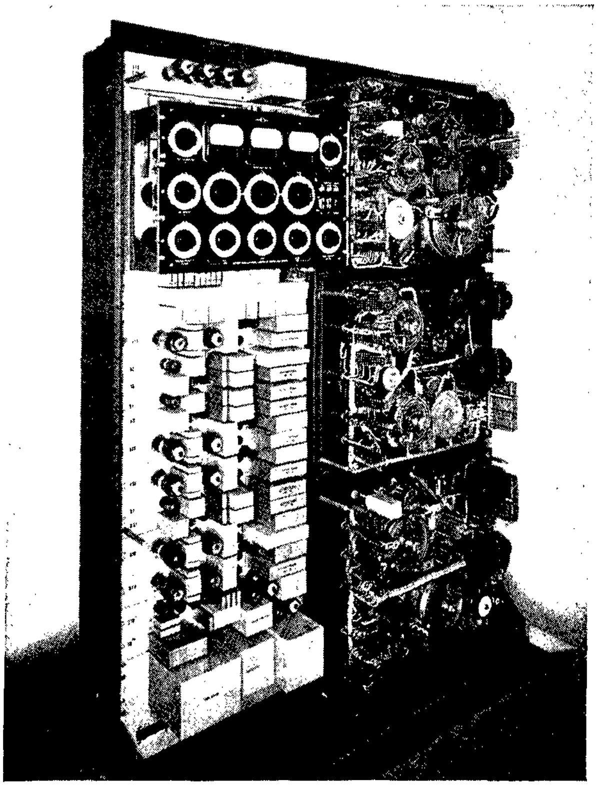


FIGURE 3. T15 front view of electrical and mechanical bays with bay covers removed.

designated T15,[3] was begun under contract NDCrc-127 with the Bell Telephone Laboratories, but most of the actual progress and all of the construction and testing were carried out under a new contract, OEMsr-353

(Project 30) with the Bell Telephone Laboratories, established especially for that purpose.

While director T10 (standardized as M9) was largely electrical in design, T15 was an attempt to combine electrical and mechanical techniques (Figure 3) using each of these techniques in such a way as to produce an optimum overall design. T15 utilized the so-called "one-plus" feature involving the addition of the prediction vector and ballistic corrections by means of mechanical differentials to the present-position vector to give gun orders. Prediction is a linear extrapolation to the future position along a straight line joining some selected past position or "memory point" to the present position. This feature eliminates the rate measuring and smoothing problems which are inherent in the design of T10. Alternating rather than direct current was used throughout.

Tests were made on T15 at Camp Davis beginning in December 1942. Although its performance was an improvement over that of M9, the difference in the two directors was masked for many types of courses by larger errors outside of the control of the director. Full-scale production of M9 was already well under way and the improvements in T15 were not considered to be sufficiently significant for standardization.

2.2.3 Curved Flight Computers

Subsequently, director T15 was modified to include fairly accurate prediction against certain types of curved courses (see Section 2.7.3). This modification was designated the T15-E1.[4] The T15-E1 (Figure 4) was based on the assumption that the pilot of an enemy aircraft would move his controls relatively infrequently. This modified director took account of the interchange of potential and kinetic energies of the airplane in a diving or climbing course. In theory, the T15-E1 would predict perfectly if the pilot of the craft maintained a constant climb or glide angle and held all his controls in a fixed position, and provided that the absolute motion of the plane in space, which is the sum of its motion with respect to the air mass, plus the motion of the air mass with respect to ground, shall be that which can be accommodated by the computer.

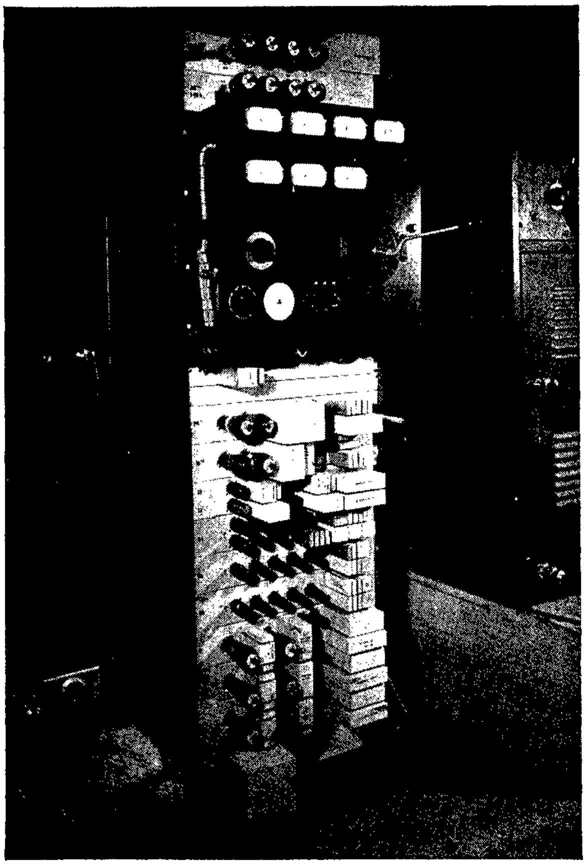


FIGURE 4. AA director T15-E1, showing control panel of curved flight bay with cover removed. The "joystick" controls rate of turn and angle of dive. Throttle and drag controls are to left of joystick. Meters in upper view are used for control of curved flight predictor. Meters in lower row are used for supervision of "memory point" predictor.

This director was compared in a series of tests with two other types of curved course directors: one a modification of M9 to be discussed below (Project 78), and another, involving a plotting board, developed by the Ordnance Department. The plotting board, designated the T12, consisted of a surface on which pens traced the present and predicted positions of the target, assuming a constant acceleration which was introduced manually; an observer watching the two pens attempted to make the future-position pen predict the course of the target by adjusting the acceleration inputs.

2.2.4 Development of Second-Derivative Curvature Attachment for M9 (Project 78)

The other attempt to include curvature in the prediction of an antiaircraft director referred to above is represented by a different and perhaps simpler solution to the curvature problem and is designed as a plug-in modification of director M9. This work was done at the Bell Telephone Laboratories under contract OEMsr-1263 (Figure 5).

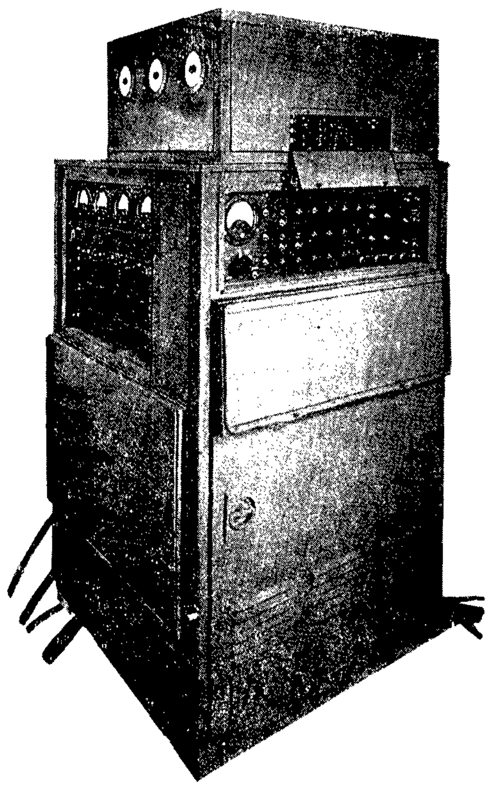


FIGURE 5. Front view of second-derivative computer.

The Bell Telephone Laboratories suggested that the linear prediction mechanism could be modified by the introduction of factors proportional to the second derivatives of the rectangular coordinates of the target course. The M9 was thus adapted so that it would measure the acceleration of the target and predict along the path in which the acceleration in each rectangular coordinate remained fixed in both direction and magnitude. One rather obvious disadvantage with this arrangement is that a constant angular acceleration in the heading of the target does not result in constant acceleration in the rectangular coordinates. Under Project 11 to be reported in Section 2.3.3, the theoretical considerations involved in the smoothing problem and in the accelerations involved in the second-derivative application were thoroughly investigated. Also, a typical but smooth target course was laid out to large scale and a prediction based on these data was made with the help of a relay interpolator to be reported in Chapter 5 (Project 70). This was done first with linear prediction, then with second-derivative, and finally with third-derivative prediction and the results were compared. The scores for the three types of prediction indicated that second-derivative curved flight prediction would improve antiaircraft fire. The reader is referred to the final technical report under the contract for further details.[5]

2.2.5 Comparative Testing

All three curved flight directors were tested at the Bell Telephone Laboratories[6] by means of courses (over 200) on a tape dynamic tester with tapes cut to represent several measured airplane courses, as well as theoretical courses. The courses generated by the dynamic tester and the future positions predicted by the three computers were recorded by the T12 plotting board. Further tests were run at the Antiaircraft Artillery Board at Fort Bliss. It was concluded that while curved flight prediction offered some hope of improved antiaircraft performance, the disturbance of the prediction by errors in the tracking seriously limited the improvement. It is possible that better smoothing means and more accurate radar data would improve the performance of curved flight predictors, but the termination of World War II halted any developments in that direction.

2.2.6 Modification of M7 Director for Field Conversion (Project 51)

As a result of experience gained in the analysis and design of smoothing circuits in connection with the development of the electrical directors at the Bell Telephone Laboratories, new and much more effective types of circuits were devised for smoothing the fluctuations of input signal to an antiaircraft predictor.

Contract OEMsr-791 was accordingly negotiated with the Bell Telephone Laboratories with the object of designing a piece of auxiliary equipment to be installed on the standard M7 director (Figure 6), embodying this smoothing technique applied to the rectangular coordinate rates. The smoothed rates were inserted in the M7 director through the wind mechanism.[7] In order to do this, a mechanical clutching arrangement had to be inserted in the shafts which supply the prediction multipliers so that the only rate input to these multipliers would be the one supplied through the wind dials; i.e., the smoothed rates.

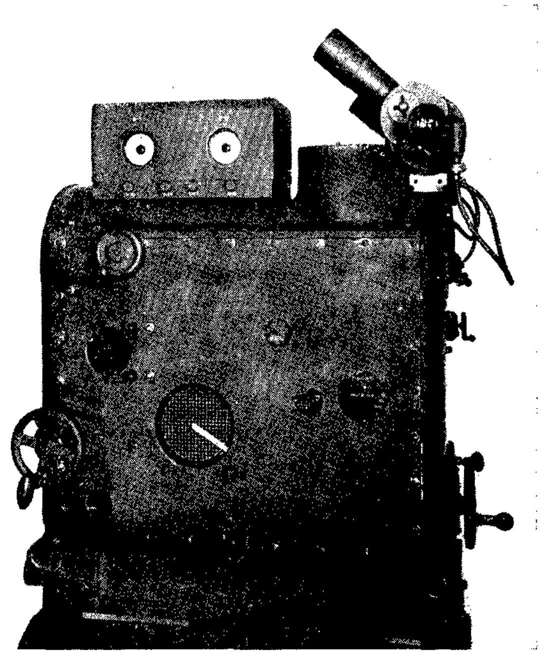


FIGURE 6. Side view of M7 director with production version of smoothing attachment rate matching mechanism (K5-9051).

A parallel development was carried on by the Ordnance Department at the United Shoe Machinery Corp. Both developments were considered satisfactory for standardization, and a recommendation to that effect was made by the Antiaircraft Artillery Board. At that time it was NDRC's understanding that this device was being put into production, but it appears that none was produced.

2.2.7 Smoothing Device for Employment with Radio Set SCR-268 (Project 64)

Under Ordnance Directive OD-122, contract OEMsr-899 was let with the Bristol Company (Project 64) to develop a device by means of which SCR-268 could be used more effectively with an M7 director (Figure 7A).[8] SCR-268 supplies radar data on the position of a target but gives rather poor values with large and violent fluctuations. M7, on the other hand, does not perform well unless it is supplied with fairly smooth data, since its amplification factor is high for frequencies of the order of those in the fluctuations. Furthermore, the present-position errors of SCR-268 are large.

Present azimuth, present elevation, and present height, which are received from the SCR-268, are resolved into rectangular coordinates and plotted. The apparatus includes a set of three charts (Figure 7B) on which the rectangular coordinates of the target are plotted automatically as a function of time. The operator matches a line to the best average of the plot, giving, through a synthesizer, smooth values of A_0, E_0, $\dot{H}_0$, $\dot{H}$, $\dot{x}$, and $\dot{y}$. The first three are transmitted to the M7 director and the others indicated at the plotter. Only the first three can be used unless the director has been modified so that rectangular rates can be set in by hand. Directors used in the United States were not so modified, although NDRC undertook a development (and the Ordnance Department undertook a parallel development) for so modifying the directors. The NDRC development was under Project 51, reported below.

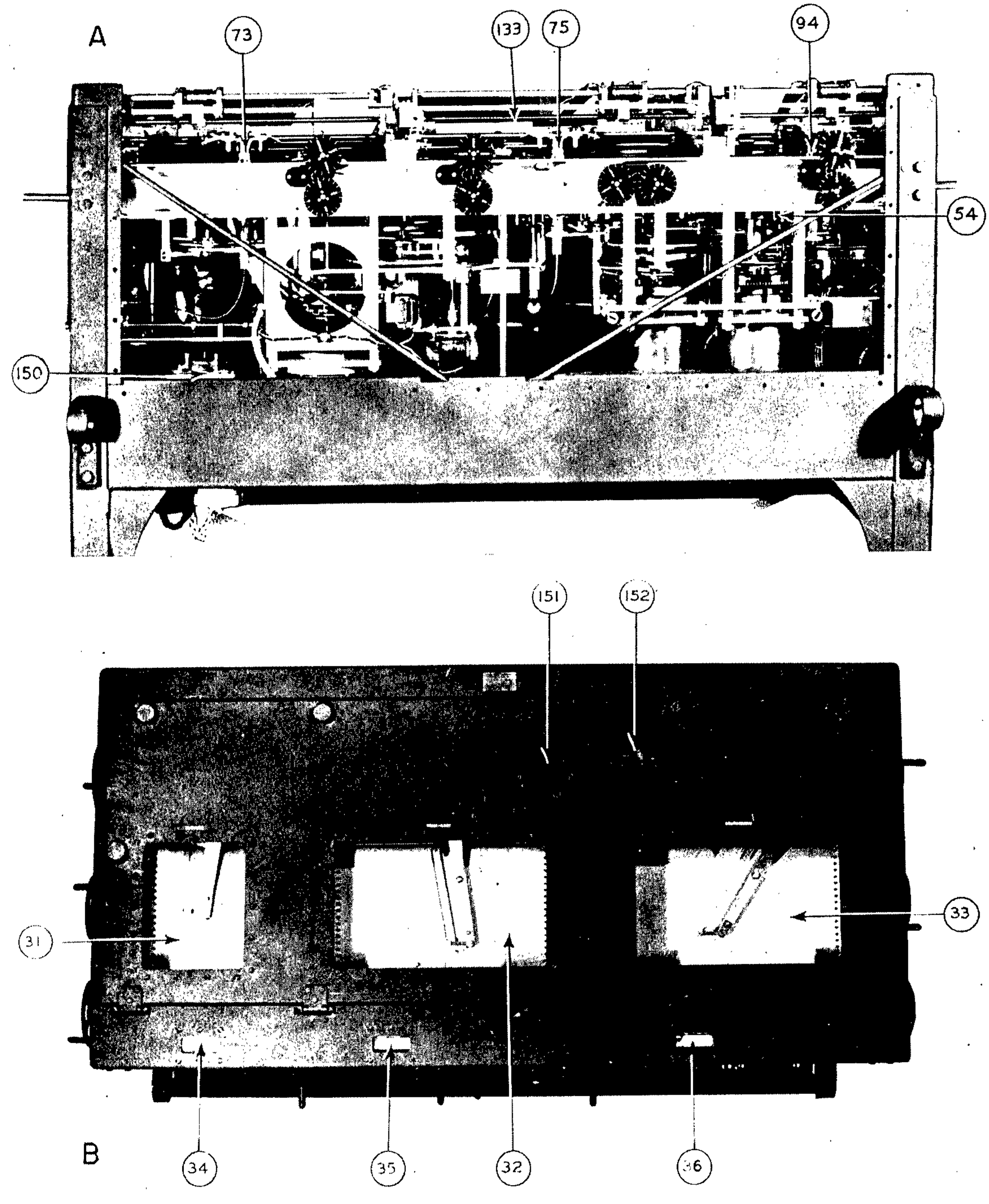


FIGURE 7. A. Side view of T1 smoother, with covers removed. B. Top view of T1 smoother showing wax-coated charts on which are plotted each rectangular component of target position as function of time (chart motion).

Means to correct the parallax between SCR-268 and M7 is also provided. The smoother was delivered to Camp Davis and was quite effective and useful for its original purpose; i.e., to smooth SCR-268 data for use with M7. By the time the testing was completed SCR-268's were being rapidly replaced by SCR-584's, the output of which was accurate and quite steady. Trials of the smoother with SCR-584 and M7 showed that the results with and without the smoother were about the same, and as a result no further work was done on this item.

2.2.8 Plotting Board T9 (Project 64)

Under Ordnance Directive OD-132, a problem was set up as Project 64 with the Bristol Company for the development of a semiautomatic plotting board (Figure 8).[9] Inasmuch as

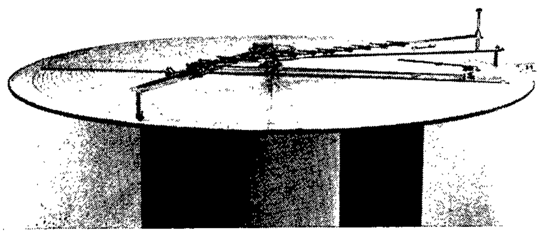


FIGURE 8. View of the ("Tenney") plotting board T9.

this plotting board was not standardized and the necessity for a plotting board was probably eliminated by the excellent combination of the SCR-584 and the M9, the reader is referred to the final report under the project for a description of the apparatus.

2.3 FUNDAMENTAL STUDIES

2.3.1 Introduction

The six projects (Projects 4, 11, 12, 17, 48, and 68) devoted to this grouping of Section 7.1 activity were initiated prior to the reorganization of Section D-2 as Division 7, and two were terminated prior to that date (December 7, 1942). Since the purposes of the various projects under this heading were conceived early in NDRC history their lines of activity are not so clear as in later projects which fell naturally into the organization of work already under way. These are among the pioneering projects of the Fire-Control Division. In fact, the work under Project 17 with the Eastman Kodak Co., contributed not only to fundamental studies, but to other activities of Section 7.1 as well, and also to the activities of Sections 7.2, 7.3, and 7.4. It approached the status of a Division 7 central laboratory in fire-control matters.

Project 17, together with Project 11 with the Bell Telephone Laboratories, was notably successful. The latter, contributing so materially to the Division 7 program as it finally evolved, and certain of its results are such an important contribution to the development of fire-control instruments generally, that a treatise has been written covering these aspects of the project, and is the subject of Part II of this volume.

Unlike the other projects of the section, those just enumerated did not contribute to what might be briefly called the production of hardware. They were set up with the intention to explore fields which might lead to devices for improving antiaircraft fire control. As might be expected, in some cases such exploration led to the development and design of equipment — "hardware" — and in other instances results were negative in the sense that it did not appear worth while to carry the project through to the stage of a finished mechanism. The production of improved hardware is, of course, of vital importance and as such is a tangible thing which is relatively easy to assess. However, the value of developments which produce only additional information, and that sometimes of a negative character, is not always minor. This the reader must judge for himself.

2.3.2 Geometrical Predictor Design at CIT (Project 4)

It was the original expectation of Section D-2 that the California Institute of Technology under contract NDCrc-164 (Project 4) would conduct a broad survey of mechanical and other units capable of carrying out the various mathematical operations used in fire-control directors. The interest of the group, however, led them to work on geometrical-type predictors.

The course of any target traveling in a straight line, together with the director location, determines a slant plane; and the prediction problem is a two-dimensional one within that plane. If one can determine that plane geometrically, lay out the line of flight in that plane, and directly extrapolate the distance traveled in that plane in proportion to the time of flight, the result will be the true future location of the target.

The California Institute of Technology group submitted a predictor design of this type, in which the target motion is resolved into a slant plane, and the computed lead angle in that plane set forward in the main geometrical representation. Their system determines the slant plane by means which are partly manual and obtains the speed of the course in that plane by means of a speedometer wheel.

The slant plane system in itself is a good approach, but it is not new; the mechanization suggested does not seem to be very practicable, and it would be difficult to achieve the necessary accuracy.

The project was terminated after submission of the preliminary report.[10]

2.3.3 Fundamental Director Studies at the Bell Telephone Laboratories (Project 11)

During the development of the T10 electrical gun director by the Bell Telephone Laboratories, it became apparent that there were many alternative electrical methods for solving the different problems which arose. Contract NDCrc-178 was accordingly initiated by the section in February 1941, with the Bell Telephone Laboratories, to study the alternatives more fully.[41]

Reference has already been made above to the studies which resulted in the development of director T15 and the relatively simple addition to the M9 computer as a prediction mechanism for certain types of curved flight. A primary outcome of this study was the emphasis it placed upon smoothing in prediction theory. This result is deemed of such importance in the work of Division 7 that a treatise on this subject, which is here published for the first time, forms part of the technical report under the contract. (Part II of this volume.)

2.3.4 Analytical Study of Prediction Devices and Construction and Tests, Iowa State College (Project 12)

Contract OEMsr-165 (Project 12) was initially concerned with prediction theory and director design. However, the particular type of prediction proposed turned out to be too inaccurate. Several other interesting ideas on director design were suggested but it was decided not to develop these ideas further.

In addition, this group did considerable work on tracking experiments. They constructed electrical analogues of mechanical aided tracking circuits. Their apparatus was constructed so that straight line courses of different speeds and crossover distances can be tracked, using different proportions of direct, velocity, and acceleration tracking, and using different types of controls, such as ordinary handwheels, small knobs, and double knobs. The use of small knobs was suggested by the hypothesis that perhaps more accurate tracking could be attained through the use of the fine muscle control of the fingers rather than the more gross musculature of the arms or back.

The project was terminated after a report of these tracking experiments was received.[11,12]

2.3.5 Electronic Computing Devices for Predictors, RCA (Project 48)

Under an earlier contract with the Navy, the RCA Manufacturing Co. carried out some work on the use of electronic computing techniques for fire-control purposes. The Navy asked that NDRC take over this work and coordinate it with other projects on electronic computers. Conferences were held between the electronic computer groups of RCA, Bell Telephone Laboratories, Eastman Kodak Co., The Massachusetts Institute of Technology, and the National Cash Register Co., and as a result

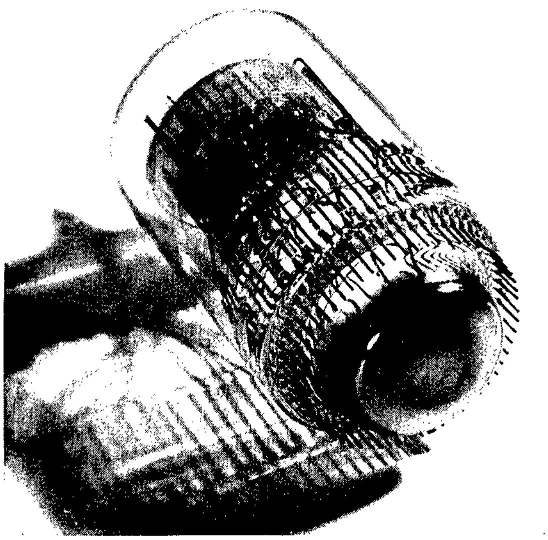


FIGURE 9. The computron.

contract OEMsr-591 was negotiated with RCA for this purpose.

Considerable progress was made in studying computing devices in which variables are represented by discrete impulses, by unidirectional voltages or currents, by alternating voltages or currents (amplitude only), and by use of the phase angle of an alternating voltage or current. Experimental units were built to carry out the arithmetical operations of addition and multiplication, and a unit started for the purpose of generating a function of two variables. Using standard tubes, the computing mechanism becomes so bulky as to be impractical. RCA designed a special multiple beam tube, called the "computron,"[13] the use of which reduces the number of tubes needed for computing by a factor of 100 or more (Figure 9). Development of this tube and a computer using it was the principal purpose of this contract.

The division decided not to extend this contract beyond the date of its termination. It became clear that there was almost no possibility of this development to reach the stage of field use in a reasonable time. Furthermore, the latest directors had their errors reduced to a point where further reduction was of questionable value until other factors affecting dispersion could be improved.

2.3.6 Mechanical Director for 90-mm Antiaircraft Guns, Bryant Chucking Grinder Co. (Project 68)

Development under contract OEMsr-1137 with the Bryant Chucking Grinder Co. was originally contemplated for the purpose of producing a mechanical director for the 90-mm batteries which would be less complicated than the electrical director T15 (see Sections 2.2.2 and 2.2.3) but which would have accuracy greater than that obtained in previous mechanical directors. Practical considerations such as simplicity of operation, ease of maintenance and repair, were to be important considerations. The so-called "one-plus"[a] method of handling the lead and the memory point system of obtaining target rates were to be given serious consideration. It was also expected that curved flight prediction would be included in this new mechanical director.

The Bryant Chucking Grinder Co. undertook the project with the understanding that initially they would complete only a fundamental study of the various problems involved, and present recommendations either for or against building the director therein contemplated. The study was not completed at the termination of World War II but the contractor's final report contains many interesting and original ideas in connection with the prediction, smoothing, and ballistic problems.[14]

2.3.7 Fire-Control Research at the Eastman Kodak Co. (Project 17)

A fundamental proposition always before Division 7 was the question of sponsoring a central laboratory, which, under direct division guidance, would investigate fire-control matters

[a] The method is called one-plus because the values computed are only those to be added to present position coordinates and not the total values.

of diverse nature. Two Division 7 projects approached this status, one at the Franklin Institute under Section 7.2 sponsorship (see Volume 3), and another at the Eastman Kodak Co. The latter contract was particularly broad and concerned work of Sections 7.1, 7.2, 7.3, and 7.4. Because of the variety of work done at the Eastman Kodak Co. under contract OEMsr-56 (Project 17), a general review of the work is given here, rather than an outline of the work done on fundamental studies only, but with particular emphasis on Section 7.1 activity. The reader will be referred to the Summary Technical Report of the other sections when appropriate.

This contract which ultimately led to what amounted to the provision of general laboratory facilities for the investigation of a wide variety of fire-control problems was initiated in June 1941.

Initially, two kinds of activities were contemplated: (a) assistance on specific optical and photographic problems which arose in the course of other investigations by Section D-2, and (b) some long-range general studies of the application of light (whether visible or invisible) to problems of detection, prediction, and servo control. Before long a considerable part of the contractor's effort was devoted to (c) general range finder developments sponsored by the group that was later to become Section 7.4 (see Volume 2). Still later, there was a large amount of development on four specific subjects, in addition to those already outlined; the first three were the responsibilities of Section 7.1 and the fourth of Section 7.3.

1. Illuminated sight Mark 14.
2. Stereoscopic observation instrument T7.
3. Intermediate range director for the 40-mm gun.
4. Applications of gyros with pneumatic restraint and take-off to various fire-control instruments. (See Chapter 4.)

Optical and Photographic Problems

Among activities of class (a) were (1) modification of the phototheodolites at Fort Monroe to give increased precision, (2) design of tracking cameras for the study of azimuth and elevation errors, (3) design and construction of an acuity testing instrument, and (4) construction of reflecting end windows to permit ranging on the sun with optical height finders. (See Volume 2.)

Light as a Working Medium

In class (b) studies were undertaken by Section D-2 along the following lines: an extended base (2- or 3-station) range-finding system;[15] an optical servomechanism;[16] and several unit mechanisms for use in a director, such as an optical memory device, an optical pointer matching device, an "optical cam" mechanism, and a high-speed electronic multiplier.[17]

General Range Finder Developments

In class (c) considerable preliminary work was done toward improving the large (13½-ft) precision range finder,[18] but early in 1943 this work was taken over by an Army-Navy-NDRC steering committee for the "super range finder," undertaking, in cooperation with the Eastman Kodak Co., Keuffel & Esser, and Bausch and Lomb Optical Co., a basic redesign of such range finders. This activity will be reported by Section 7.4. (See Volume 2.)

The majority of the range finder work of Project 17 has been in the development of a number of short-base range finders with autocollimating features, and mostly of the superimposed coincidence type. The first one, which served as a prototype for the others, was a 15-in. superimposed coincidence instrument.[19,20] This was tried in several modifications for various purposes, including tracer crossover observation for machine gun sight adjustment both in plane-to-plane fire and ground-to-plane fire.[21] For various reasons none of these was successful. A number of the 15-in. instruments were delivered to the Navy, principally for trial in connection with the Navy's gunsight Mark 14.

A second embodiment of the superimposed coincidence principle is the 30-in. range finder for feeding range into directors of the intermediate type.[20] After several modifications this emerged as an instrument of 30-in. base length and 8 power, with color differentiation between the two images. The ranging mechanism is interconnected with the elevation handwheel of the director so as to make it belong to the "height finder" rather than to the "range finder" class. It is provided with two eyepieces, one of which is used by the ranging operator to keep the images matched, and the other by an observer who watches tracer crossover and introduces the necessary spots into the director. This instrument was incorporated in director M5A1E1 (Project 31) and tested by the Antiaircraft Artillery Board at Camp Davis. The results were especially promising and, hence, six additional directors were equipped, delivered, and field tested. The director including this range finder (now standardized as M10) has been standardized as M5A2 and a large quantity was built.

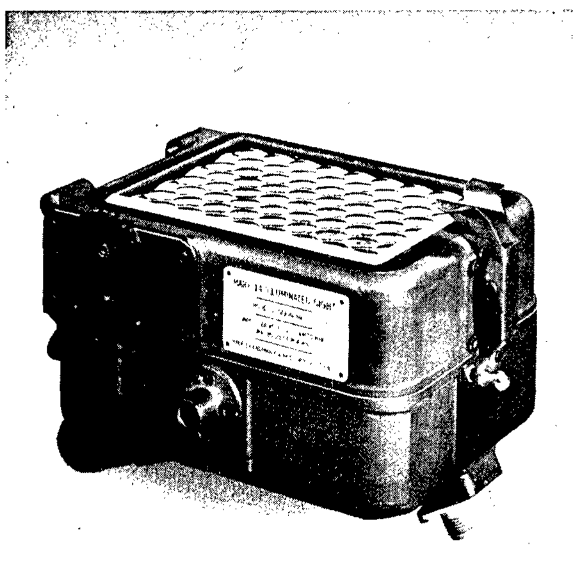


FIGURE 10. The "fly's eye," Mark 14 illuminated sight without 45-degree reflector plate.

Two 48-in. range finders, almost identical with the 30-in. director instrument mentioned above, were constructed for use with the intermediate range directors.[20] One of these is installed on director T28 developed under this contract. Other models were designed, particularly for infantry use, but were not standardized, although two models (designated T25 and T26) were constructed and extensively tested before, during, and after several modifications. T25, as finally modified, appeared very promising and nearly ready for standardization.

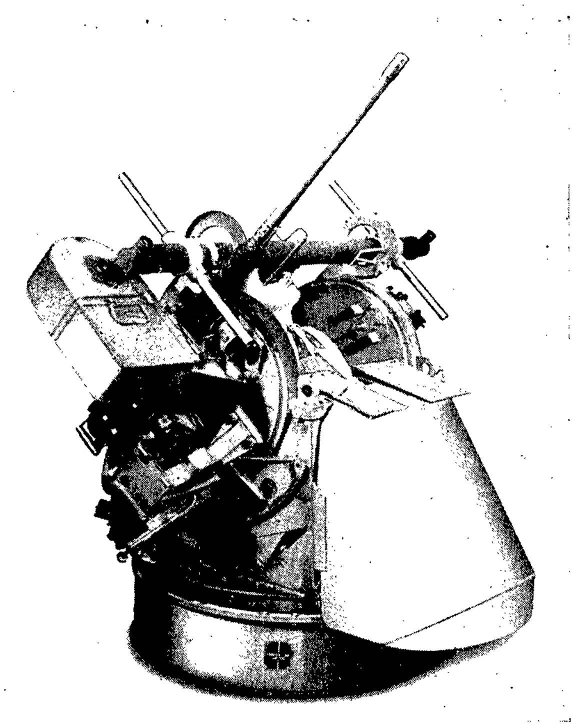


FIGURE 11. Stereo aid for Maxson turret.

ILLUMINATED SIGHT MARK 14

Under subject 1 was developed a reflecting sight known as the "fly's eye" (Figure 10) because of its multilens nature, the advantage of which was that the freedom of the gunner's head movement was greatly increased.[22] This will be reported in more detail by Section 7.2 in Volume 3.

STEREOSCOPIC OBSERVATION INSTRUMENT T3

Under subject 2, a stereoscopic aiding device has been developed for increasing the accuracy of tracer fire control with automatic weapons.[23] Considerable work was done on this subject by the Polaroid Corporation under Project 32 (see Section 2.6.1). The first model by the Eastman Kodak Co. was unsatisfactory, chiefly because of the combined effect of low light transmission and excessive vibration. The Eastman Kodak Co. constructed another model employing a new optical system which showed promise of being much better under the excessive vibration present. This model was tested at Fort Bliss and proved to have two difficulties. The most serious difficulty was that the exit pupil was only about 2 mils larger in diameter than the usual eye pupil size. Despite all the work done to eliminate vibration there remained some vibration in the instrument as well as some in the observer's head. The other difficulty was due to an error in the optical system which resulted in some curvature in the field. Another model was constructed with large exit pupil and a flat field. The results of firing trials were very satisfactory with range up to about 700 yards, beyond which the stereo perception rapidly became unsatisfactory. Several small OQ planes were shot down with the .51-caliber 4-gun turret M1 at 400 to 600 yards (Figure 11).

Intermediate Range Director for the 40-mm Gun

Under subject 3 an intermediate range director T28 (Figure 12A) was developed.[24] One of the Eastman engineers suggested the use of a sphere with three mutually rectilinear dipole windings[25,26] as a combined resolver, lead computer, and synthesizer (Figure 12B). The sphere is turned by present coordinates, and the leads inserted simply by suitably energizing the dipoles. The proper locations for the gun azimuth and elevation axes are thus determined (Figure 13). A serious difficulty in determining the instantaneous rates was encountered. At NDRC's suggestion, therefore,

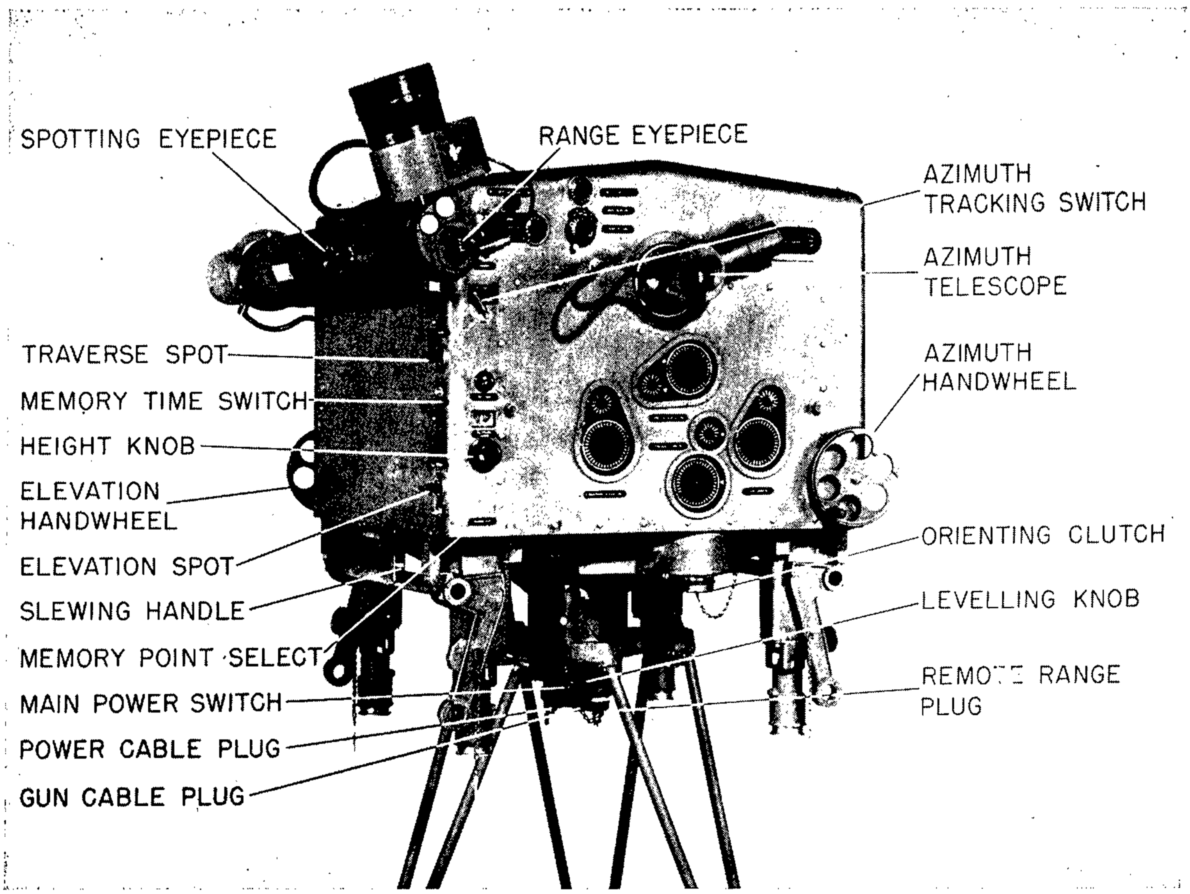


FIGURE 12A. T28 director.

the attempt to obtain instantaneous rates was abandoned, and the project proceeded on the memory point system.

The preliminary tests were completed at

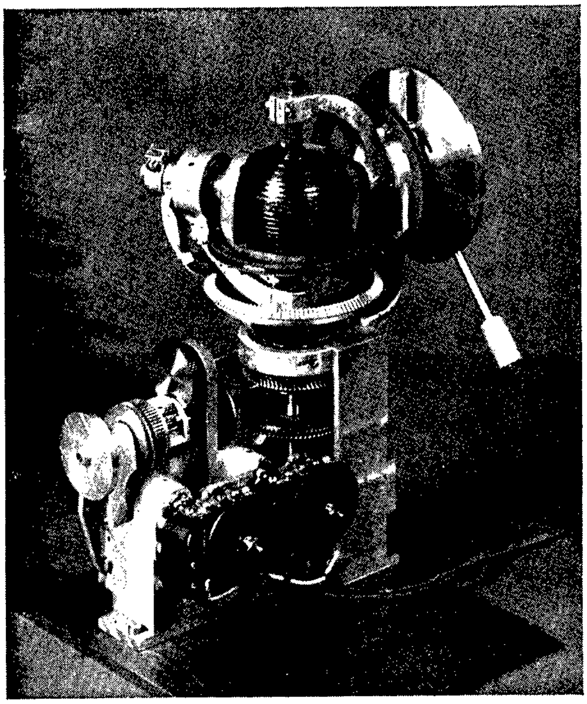


FIGURE 12B. "Magic Ball" or combined resolver, lead computer, and synthesizer making use of three mutually rectilinear dipole windings, wound on sphere.

Fort Bliss, Texas, and disclosed a number of troubles and errors that needed to be corrected, but the termination of World War II stopped further development.

PNEUMATIC CONTROLS

Under subject 4, Section 7.3 carried out a series of considerable developments of pneumatic computing systems, which will be reported in more detail by that section. (See Chapter 5.)

2.4 AUTOMATIC WEAPONS

2.4.1 Development of Intermediate Director M5A2

The third category of Section 7.1 activity listed above (see Section 2.1.2) was the development of automatic weapons. This work was in reaction to the very unsatisfactory state of 40-mm fire control. It cannot be said that a complete and satisfactory system had been developed even at the termination of World War II. At the beginning of the war the standard director for control of 40-mm fire was the M5 (Kerrison) director which depended upon certain observations of the tracers, the condition for such observations being very difficult to obtain in the field, particularly against high-speed targets. Consideration was given early in 1941 to this problem and several steps were taken at various periods to improve this situation.

2.4.2 Barber-Colman Co. (Project 31)

Contract OEMsr-268 was instituted at the Barber-Colman Co. which sought to provide a better working, simpler, and more easily procurable modification of the M5 director. The principal substitutions were eddy current drag disk governed motors for the ball integrators and torque amplifiers, and an electronic multiplier for the mechanical multipliers. The director, T21, was completed, but testing at Camp Davis showed no improvement in hits, and the saving in cost was judged insufficient to warrant changing. Furthermore, the most troublesome feature of M5 had meanwhile been cured by the elimination of the torque amplifiers and substitution of larger and higher speed ball integrators as the variable speed drives.

In order to try any improved solution, it was necessary to have a servo for the 40-mm gun which would follow with reasonable accuracy. The existing servo had a very bad velocity lag —as much as 2 degrees at high speeds. The contract contained authority to build a better servo, and a clutch-type unit was constructed and used for many of the tests.

As the next step the Eastman 30-in. red-green range finder developed under Project 17 was mounted on the director. This range finder provides the tracer crossover method of spotting. This combination did not give satisfactory results, principally because it was difficult to train a ranging operator to follow the rapidly changing range. Aided tracking was provided, but not only did the range change rapidly but also the rate at which it changed. Also, the operator could not satisfactorily range and adjust the fire at the same time.

modified M5 was built mounting the Eastman range finder and having means by which range would be automatically driven into M5. With this modification spotting in range was easily

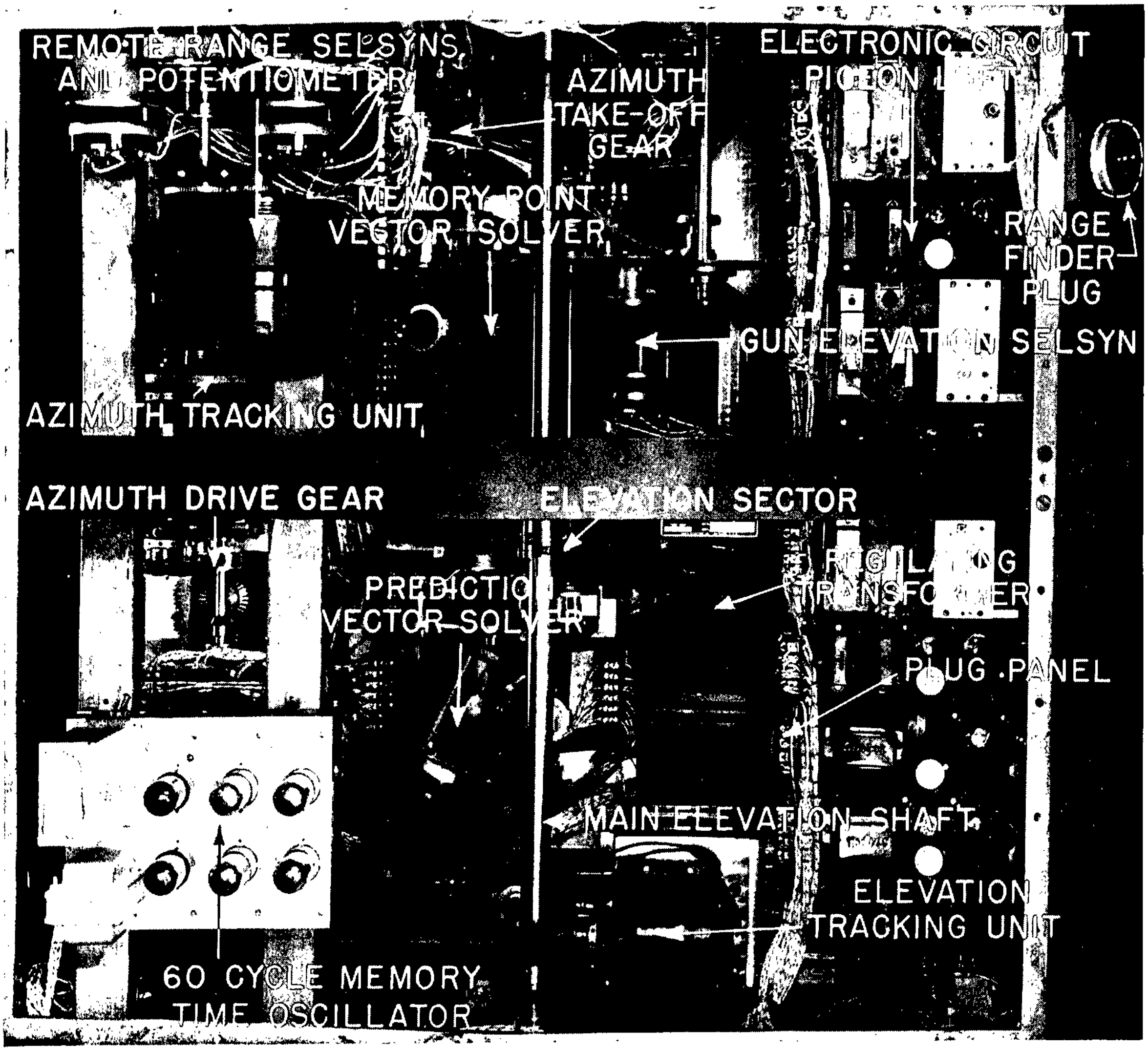


FIGURE 13. Interior view of T28 director.

Two changes were then made, one to permit the operator to match range by setting height instead of slant range and the other to provide two eyepieces so that a second operator could adjust the fire. Angular height was obtained from the director and slant range automatically computed and power-driven into the director.

Inasmuch as it was desirable to convert M5's rather than build completely new directors, a provided and the method of operation was very similar to that employed with M5. The differences (and they were very important ones) were: nearly correct range continuously supplied to the new director designated as M5A1E1,[27] and an opportunity to see on every shot the direction and amount of necessary correction, whereas with M5 one can only tell the direction of correction, and then only on rare occasions when a line-of-sight shot is obtained.

Six M5A1E1 directors were delivered to the Army on June 20, 1943. These six units were sent to the South Pacific, and very favorable informal reports of their performance were received. The director was standardized as M5A2, and the production order was given to Singer Manufacturing Co. (Figure 14).

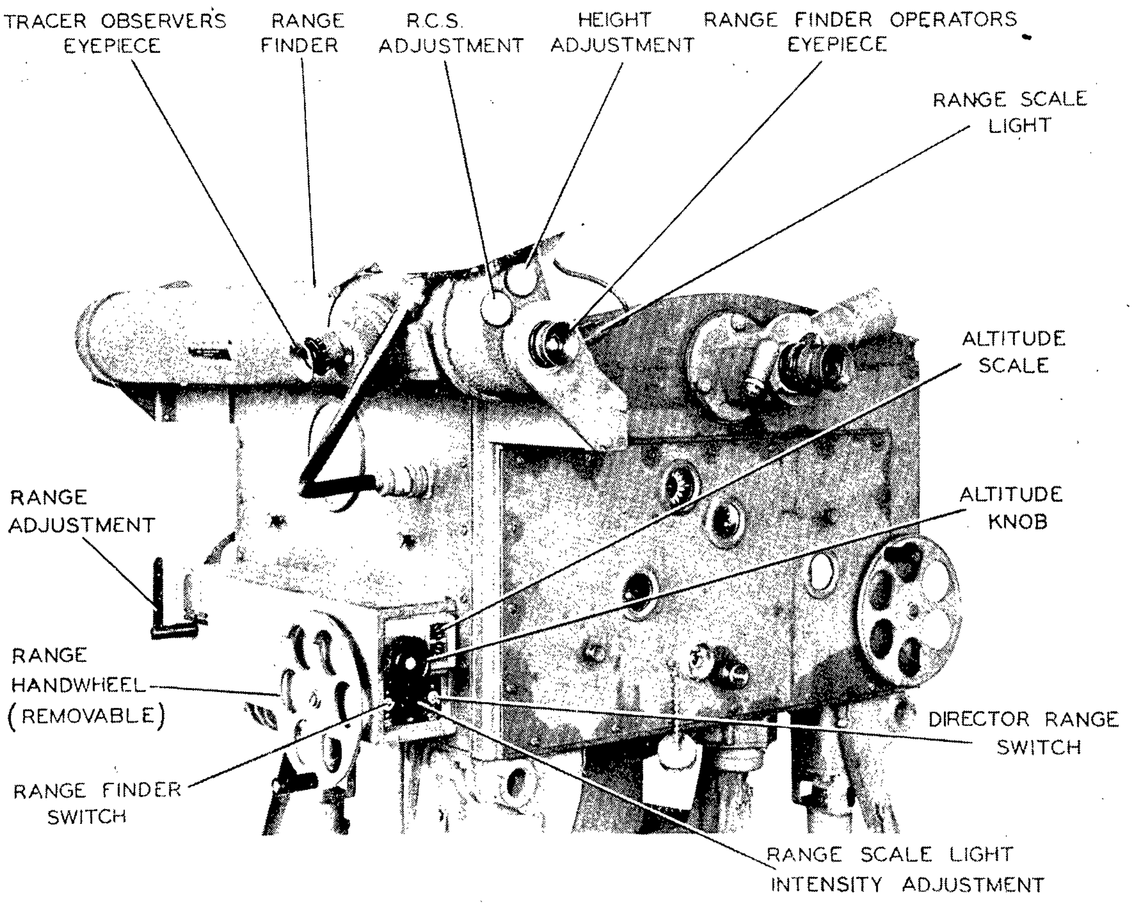


FIGURE 14. M5A2 director.

The Antiaircraft Artillery Board requested that a model be supplied them for test embodying modifications to see whether spotting in elevation and azimuth is feasible by means of a joystick hand lever. With this arrangement both trackers continuously track on the target. The model including the joystick hand lever was tested but proved to be little or no improvement, for the spotter had difficulty in knowing the exact direction in which to push the lever.

Another modification having separate azimuth and elevation spotting arrangements was also tested and appeared to be as good as, or possibly a little better than, the standard range spotting used on M5A2. It is doubtful that the improvement is great enough so that the Antiaircraft Artillery Board will recommend a change.

The Antiaircraft Artillery Board also requested that a standard director M5A2 be modified so that it would accept radar inputs. The director was delivered to Fort Bliss in June 1945, but when supplied with data from SCR-584, it was found that the perturbations of SCR-584 data at about one cycle per second were considerable and that a smoother had to be introduced between SCR-584 and M5A2.[28]

A series of mathematical studies[28] on gyroscope substitutes were carried on under this contract for comparison with standard gyroscopes; one was a fluid gyro and the other a vibrating reed tachometer. Both proved of interest but of insufficient advance over standard gyros to warrant an intensive program. (See Chapter 3.)

Two other projects contributed to the development of M5A2: the Project 17 (Eastman Kodak Co.) contribution of a suitable range finder cited above, and an experimental investigation carried on at the GM Laboratories, Chicago, Illinois.

2.4.3 GM Laboratories, Inc. (Project 26)

Under contract OEMsr-184 the GM Laboratories was requested to develop an electrical lead-computing device for intermediate range directors, and a servomechanism for converting the lead angles from the electrical to the mechanical form was accordingly undertaken. By the use of suitable tapered potentiometers, and with a stable electronic amplifier and induction motor combination, the desired accuracy of plus or minus 0.5 per cent between the input and output rotary motions was obtained. A model incorporating both elevation and azimuth systems was built and with some modifications was used in the standard director M5A2.[29] (See Chapter 3.)

2.4.4 Development of Intermediate Director M7 (Weissight)

A schematic of a course-and-speed emergency sighting system adaptable for use with the 40-mm antiaircraft gun or the Maxson turret was drawn up,[b] and a diagrammatic model was made to show the method of operation. The system is a mechanical vector solution in which course and speed are set by estimation, but it is stabilized in azimuth so that once the course is correctly set it will remain correct so long as the plane flies in a straight line.

[b] By H. K. Weiss of the Antiaircraft Artillery Board.

2.4.5 Development at the Pitney-Bowes Postage Meter Co. (Project 61)

A preliminary model for the 40-mm Bofors was built under contract OEMsr-883 at the Pitney-Bowes Postage Meter Co. and tested at Camp Davis. As a result of comparative tests between this sight, a British version of course and speed sight known as the Stiffkey stick, the forward area sight, the Weissight was standardized as M7 for use on the 40-mm gun, built in considerable quantities, and used throughout the remainder of World War II.

With this device an additional operator is employed whose sole duty is to set course and speed. The usual two trackers are thus enabled to concentrate on following the target without diverting their attention to estimating lead (Figure 15).[30]

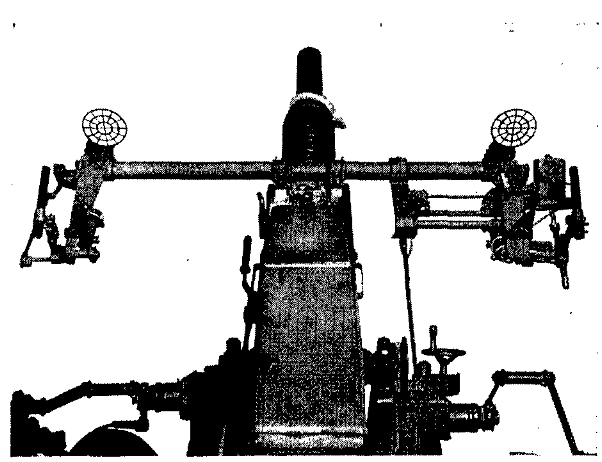


FIGURE 15. Course and speed sight, M7.

The course and speed sight is set by pivoting the arrow (located on top of the box in the extreme right of the picture) along target path, and setting the dial on the box in accordance with estimated target speed. The telescopes are then used to track the target. The gun will "lead" the line of sight in accordance with the problem by virtue of geometrical links between the gun and the telescopes. The forward area sights (cart wheels) are stand-by, or emergency, sights and should not be confused as part of Weissight system.

2.4.6 Course-Invariant Sights

Perhaps one of the most serious problems in connection with the use of gyro lead-computing sights (see Chapter 4) is that an accurate and continuous range is necessary in order to have a good solution, and such a range is not usually obtainable. Inasmuch as the rate of change of range often is rapid and its derivatives also are rapidly changing, the possibility of estimating range correctly and continuously is very slight. At present the most satisfactory arrangement appears to be to set a range through which the target will fly. This system is moderately satisfactory against directly approaching targets but is not satisfactory against crossing targets. In view of this difficulty, a type of lead-computing sight without range input was proposed,[c] by which invariants of the course can be inserted and, by appropriate mechanisms, continuous range produced for and introduced into the computation. Thus, in using such a sight, the operator might attempt to refine his estimate of invariant during tracking either by observing tracers or by comparing the range indicated in the computing mechanisms with the true range at some point of the course. The advantages to be gained by such a sight led Section 7.1 to negotiate a contract for its development.

2.4.7 Development at Baker Manufacturing Co. (Project 73)

Accordingly contract OEMsr-1190 was negotiated with the Baker Manufacturing Co. for studying methods of adapting gyroscopic sights to the invariant system of lead computation. A proposal for an invariant gyroscopic sight was made by the company and was discussed in a conference at the Applied Mathematics Panel.[31]

This development can be considered as a new approach to automatic weapons fire control which will permit one-man operation (Figure 16) and which, in very general terms, combines the desirable qualities of the gyroscopic lead-computing sight and those of the vector sight.[32] The new sight is on the carriage, and in installation and in tracking of the target more nearly resembles a gyro lead-computing sight than a separate director such as M5. In more detail, an angular lead is produced from a precessed gyro as in the gyro lead-computing sight, but the lead angle, instead of being proportional to angular velocity and to set-in range, will be such a function of the angular velocity that a one-knob adjustment once correctly made for a given unaccelerated target course remains correct for the remainder of this course.

FIGURE 16. Invariant gyroscopic lead-computing sight mounted on M45 turret (gun removed).

The optical arrangement was decided upon in conferences with the Eastman Kodak Co., and this section of the mechanism actually was built by Eastman under Project 17 referred to above.

The sight, completed after the termination of World War II, was mounted on a Maxson turret and some testing completed. Vibration troubles had been expected and an attempt made to guard against them, but when four guns of the Maxson turret were fired the mount vibrated so badly that firing tests could not be conducted. Considerable development appears to be necessary to overcome this difficulty. The principle itself appears to be promising, and it is thought that the project should be continued by the Army.

[c] Independently by H. H. Germond of the Applied Mathematics Panel and H. K. Weiss of the Antiaircraft Artillery Board.

2.4.8 Intermediate Range Director T28

Under the broad fire-control research program at the Eastman Kodak Co. (Project 17) (see Section 2.3.7), there was developed the novel intermediate range director T28 for the 40-mm gun. The most particular point of novelty resides in the use of an electromagnetic dipole vector solver. The reader is again invited to review the Eastman report[24] for details of this important development.

2.5 MISCELLANEOUS WEAPONS

2.5.1 Antiaircraft Rocket Director (Project 38)

Contract OEMsr-517 with the Bristol Company was negotiated in response to a request of the Ordnance Department (Directive OD-56) for the development of a director for controlling a battery of 3-in. antiaircraft rocket projectors. The director was to be useful against high- and low-level targets, as well as against diving targets, and provision was to be made for operation with either optical or radar inputs.

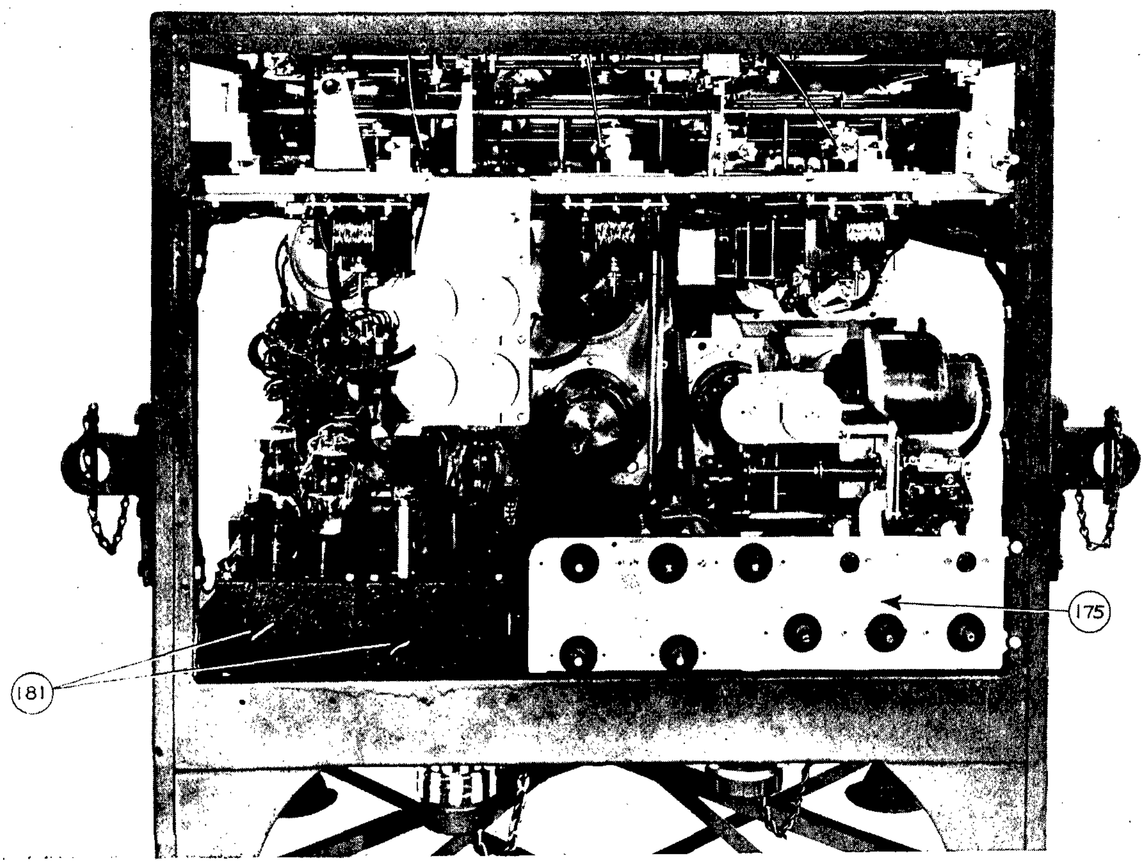


FIGURE 17A. AA rocket director T18 side view, covers removed.

The apparatus consisted of two separate parts — a tracker and a computer (Figure 17A). The tracker (M5 modified) was to be used for nearby targets, whereas the tracker and the computer was to be used for distant targets. The tracker included the usual optics and their aided drives, together with transmitting selsyns to get the data to the computer. In addition, the direct angular prediction mechanism included in the tracker was for use against near and diving targets, and the same selsyns transmitted to the projectors directly.

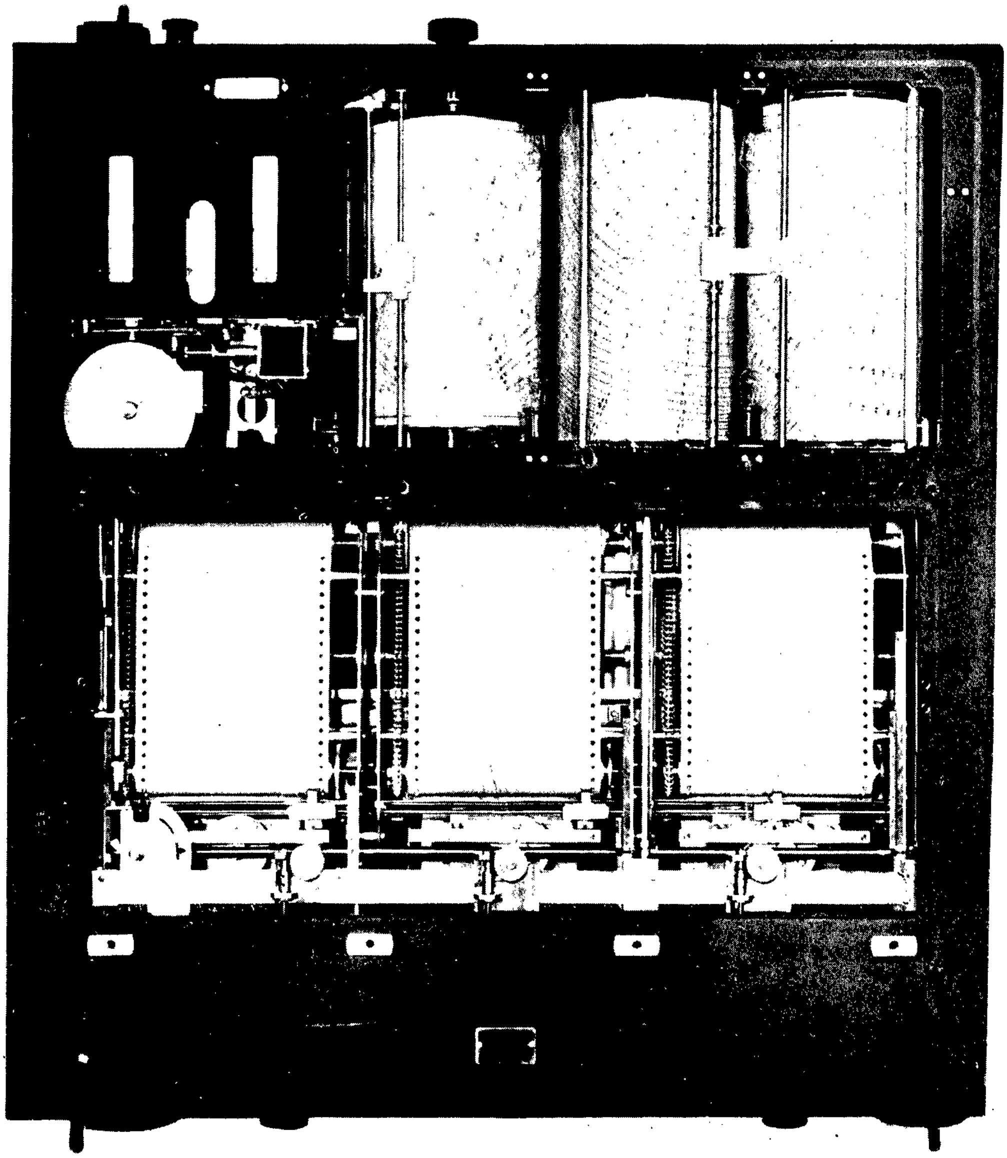


FIGURE 17B. AA rocket director T18, top view showing charts: waxed (lower), and ballistic (upper).

The computer incorporated three recording voltmeters, and rates were obtained by matching the slopes of the lines. Inasmuch as continuous fire was not contemplated no feedback of time of flight was necessary, and the solution was extremely simple. The ballistics were included, only partly mechanized, i.e., a chart reading (Figure 17B) had to be set on a dial. Selsyns transmitted the information to the projectors.[33]

The director was completed and trials at Camp Davis were finished, except for firing. No ammunition was available, and the projectors were not sent to Camp Davis. Operation of the director was satisfactory and the instrument itself was in form suitable for production.

At this point NDRC was informed that there was no longer a requirement for rocket projectors and, therefore, none for the director.

A modification of this director for use as a data smoother and retransmitter between SCR-268 and M7 was built under Project 64.

2.5.2 Antitank Computing Sight T62 (Project 59)

At the request of the Ordnance Department a lead computer for the 75-mm gun mounted on the motorized gun carriage M3 was developed at the Barber-Colman Co. under contract OEMsr-892.[34]

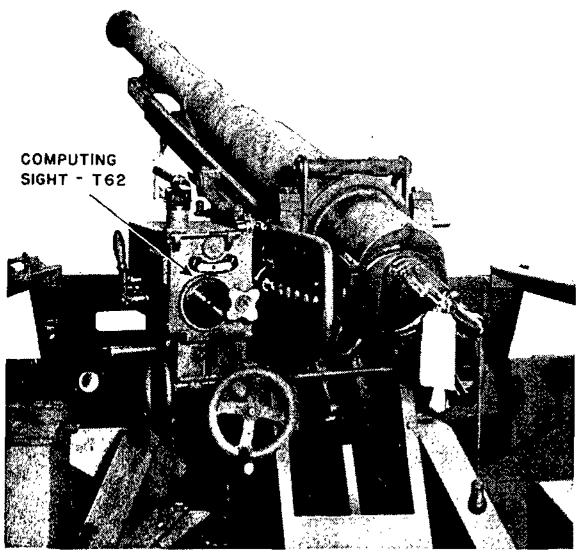


FIGURE 18. Antitank lead-computing sight T62.

The computer (Figure 18) offsets the telescope from the gun bore axis in accordance with (1) lead angle in azimuth, (2) azimuth correction due to tilt of the gun trunnions, and (3) super-elevation correction. The azimuth lead angle is computed by the memory point method, the azimuth travel of the target being measured for a time equal to the (estimated or measured) time of flight. An advantageous feature of this design is that as the lead angle changes due to change of range and angular rate, the telescope displacement is altered only by the change in lead angle. Another action of this character restores the telescope to alignment with the bore before a new computation is made.

Successful firing trials of the computing sight T62, mounted on the gun carriage M3, were conducted at Aberdeen, April 25 through May 1, 1944, and at the Tank Destroyer Board, Camp Hood, Texas, August 12 to 14, 1944. The weapon was turned over to the Tank Destroyer Board as a museum piece at the request of the Ordnance Department.

2.6 SHORT-BASE RANGE FINDERS

Reference has already been made to the range finder developments at the Eastman Kodak Co. sponsored by the division, which was one phase of a very broad contract set up with that company for the development of fire-control instruments. In the interests of integration, all activity of Section 7.1 at the Eastman Kodak Co. was reported in Section 2.3.7 with a synopsis given of the work sponsored by other sections. In addition to this work done under the section's auspices on short-base range finders two further projects are to be reported.

2.6.1 Polaroid Corporation (Project 32)

Contract OEMsr-302 was originally established for the purpose of developing short-base range finders suitable for use in connection with intermediate range antiaircraft guns and plane-to-plane fire control. Under Bureau of Ordnance Directive NO-112, a 43-in. stereoscopic range finder of very simple design was worked out and 25 were constructed for trial. A comparative test of these and the 15-in. range finder, developed by Eastman Kodak Co. under Project 17, showed the two instruments to be of about equal precision when used by unselected and untrained personnel. In the hands of carefully selected and trained personnel the Polaroid range finder was more precise.

The Bureau of Ordnance later requested that twelve 43-in. range finders be equipped with mechanism for setting range continuously into the Mark 14 lead-computing sight. The first of these devices was in combat service and the second was later completed and used for demonstration at Dam Neck on February 23, 1944. The design and construction of an improved model employing the motor servo was started, but this was discontinued because the results of the first unit in combat service indicated that there was not sufficient time to make the necessary observation, and it was difficult to keep the sight operator and the range finder operator on the same target. It was concluded that the benefits which could be derived from such a device were not sufficient to justify further work on the project.

The stereo aid project, discussed under Project 17, was initiated at the Polaroid Corporation and the first model constructed by them.

Several other ideas for short-base range finders were investigated, but the development of them was not carried very far. They are related in the Polaroid final report.[35]

2.6.2 Combined Tracking and Range-Finding Devices (Project 52)

Contract OEMsr-735 was set up with the Barber-Colman Co., contemplating the design and construction of two combined antiaircraft tracking and range-finding devices. One was to be for a 30-inch-base range finder and one for a 13½-foot-base range finder. Some work was done on each but neither was constructed; in fact, this project was changed a number of times.

The range finders which were to be used as an integral part of these tracking devices were those (self-collimating) which were developed under NDRC auspices at the Eastman Kodak Co. and reported above under Project 17.

At one time, the unit which was to have a 13½-foot range finder was to be redesigned because of an urgent need for a tracker for the director T15, the principal object being to provide a substantially lighter and cheaper tracker than the one used with M9 and combined with a range finder.[36] The potentiometers and associated mechanisms required in the M9 tracker are obviously not required for T15. The layout was substantially completed, and then all work was stopped because T15 was not standardized.

2.7 TESTING PROGRAM

At the time the NDRC started its work, one of the apparent lacks in the equipment of the Army and Navy was an adequate program for testing antiaircraft fire-control systems. Furthermore, there was in existence very little test equipment to carry out such a program.

2.7.1 Heavy Antiaircraft Fire Control

Probably the most straightforward problem was that of testing heavy antiaircraft fire control. This was true because the devices which constituted the system were largely automatic, thus making unnecessary tests involving human elements. For test purposes the fire-control system was divided into three parts, namely the tracking equipment, the predictor or computer, and the gun. The computer was selected as an appropriate subject for investigation because of its importance in the chain and because new computers were being proposed and developed.

2.7.2 Dynamic Tester (Project 25)

The first device designed particularly for testing computers dynamically (Figure 19), that is, in motion as if they were actually operating, was developed under contract OEMsr-98 with the Barber-Colman Co. This machine was outlined and proposed by the Barber-Colman Co. in order that information could be obtained rapidly about two things: first, the errors committed by a computer when the data supplied to it was essentially perfect; and second, the manner in which the computer would handle certain types of perturbation introduced by the imperfections of the tracker.[37]

The perfect data for the Barber-Colman dynamic tester was stored by cutting a set of very accurate cams. Servo motors associated with three of the cams were provided to drive the handwheels of the computer under test exactly as if a perfect tracker were following an idealized target. The correct gun orders for this target were stored on the remaining three cams and compared electrically with the gun orders actually transmitted by the computer. Errors in the gun orders were recorded electically on graph paper.

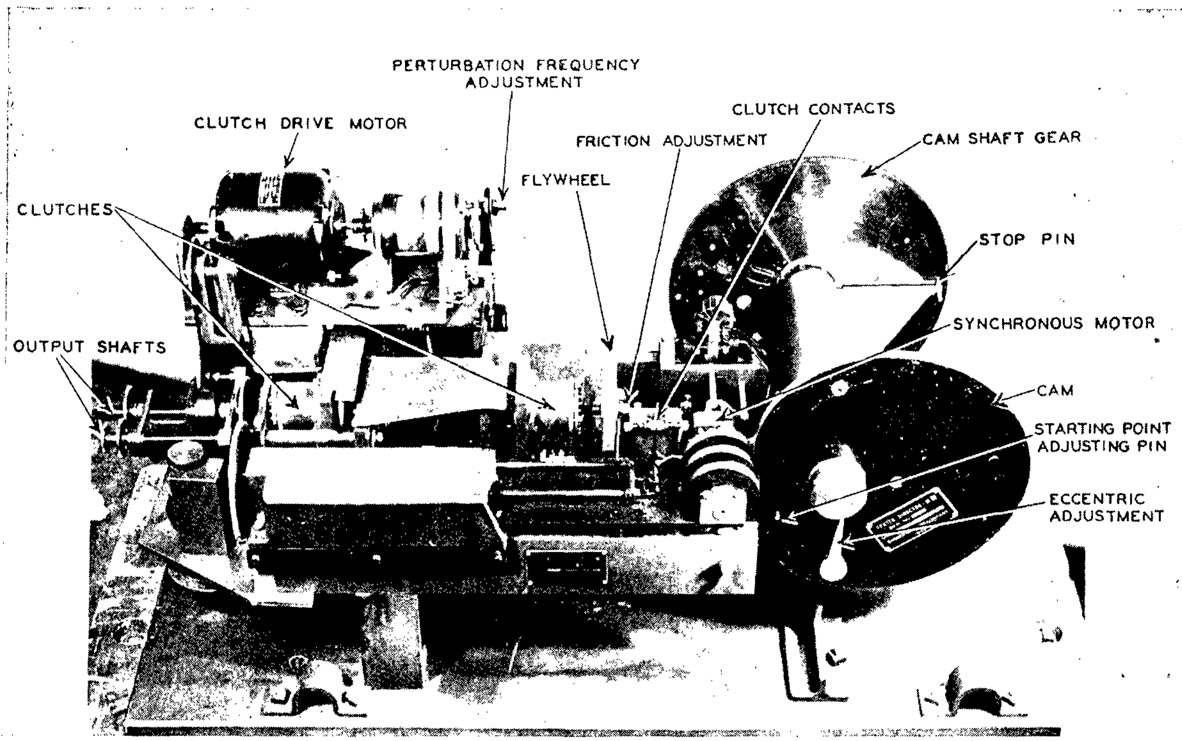


FIGURE 19. Barber-Colman dynamic tester range unit.

The dynamic tester was found to be of great value, since many runs could be made in a short time and the errors observed instantaneously. A perturbation unit added sinusoidal variations about the perfect tracking data and was so arranged that the effects of various amplitudes and frequencies could be observed. It is readily apparent how important such a rapid evaluation of errors can be in the expeditious development or modification of complicated systems.

2.7.3 Punched Tape Dynamic Tester (Project 60)

When in the later stages of World War II the problem of coping with evasive action and curved flight on the part of enemy targets was encountered, it became clear that a great variety of test courses would be necessary.

The dynamic tester developed under Project 25 was limited as to the number of courses available by the fact that each new course required a set of input and output cams, which were rather expensive. Also, the process of changing from one course to another required several hours.

To simplify the process of preparing data for use by a dynamic tester a new mechanism was outlined and proposed by Section 7.1, and contract OEMsr-904 with the Bell Telephone Laboratories was undertaken to develop a dynamic tester in which the data for courses is supplied by punched tapes which could be prepared and interchanged with a great saving in time and expense.[38]

A breadboard model of one unit of the tape tester was first constructed to prove the practicability of the idea. A complete tester, called the Model I tester, with three input and three output channels (Figure 20) was then built and used at the Bell Telephone Laboratories for the tests on curvature modification of directors T15 (Project 30) and M9 (Project 78), discussed above. Later, the Model I tester

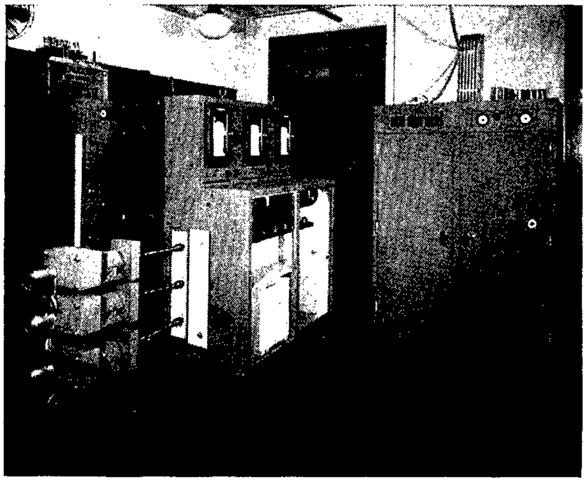


FIGURE 20. The punched tape dynamic tester—complete assembly.

was used for a number of other projects and was finally delivered to the Antiaircraft Artillery Board.

The testing of naval directors such as the new Mark 56 calls for a dynamic tester having a larger number of inputs and higher accelerations resulting from the ship's motion (see Chapter 6). A model was accordingly developed having six inputs and capable of accelerations of the order of 1,000 mils per sec^2 with a probable error of about 1 mil. It has three output channels utilizing a "pseudosynchro" capable of making spot checks of the outputs at 0.3-sec intervals, the errors being recorded on recording meters as in the Model I instrument.

A bench model was set up during the summer of 1944, and the first completed instrument was expected by January 1945. An unavoidable delay was brought about by a prolonged search for the cause of faulty operation of the bench model. The first Model II instrument was to be delivered to the Radiation Laboratory for the Mark 56 tests, but on account of changes in plans occasioned by the end of the war, this model was delivered to the Naval Research Laboratory. A second Model II was under way for the British, but the discontinuance of lend-lease made it necessary to cancel the order for this instrument.

An auxiliary device called the Relay Interpolator was built at the suggestion of Section 7.1 under Section 7.5 auspices (Project 70) to facilitate the preparation of the tapes at the Bell Telephone Laboratories, under contract OEMsr-1160. A summary report by Section 7.5 on this device will be found in Chapter 5.

Although the dynamic testers could simulate tracking data from any of the standard sources, the Army prefers to have a part of the test run with the computer receiving actual tracking data at the testing ground. When such tests are run, difficulties of recording data and of calculating the errors are such as to overload the computing facilities available for the work. Unlike the dynamic tester tests, this method cannot reproduce courses exactly. Hence much more data must be obtained in order to derive reliable results.

2.7.4 Data Recorder and Ballistic Computer (Project 63)

Mechanisms were built under Division 7 contracts to ease the situation. The first of these was a data recorder proposed by Section 7.1 and built by the Bell Telephone Laboratories under contract OEMsr-965.[39] It accepts standard selsyn data from the tracker and the computer, translates and prints these quantities in numerical form on six tapes. The data are thus immediately available. Normally a record is made at intervals of 1 sec within 0.004 sec or less and records to the nearest ½ mil or to the nearest yard. The mechanism provides for each channel a selsyn-servo combination, a gear train with commutators which set up digital codes on relays, and a standard "ticketer" or printer, together with such control relays as may be required.

To shorten the computing time a ballistic computer was built at the suggestion of Section 7.1 under Section 7.5 auspices at the Bell Telephone Laboratories under contract OEMsr-1236. This device to be reported upon by Section 7.5 (see Chapter 5) carries through automatically all the operations needed to calculate the correct times of flight and gun orders for the data observed in the tests, and prints the errors committed by the predictor. It does the work of about 50 girls with standard calculating machines.

2.7.5 Liaison with the Antiaircraft Artillery Board (Project 54)

Testing of the various antiaircraft devices developed by the Fire-Control Division of NDRC was carried on chiefly by the Antiaircraft Artillery Board. A very close working relationship was maintained between NDRC and the Board, and contract OEMsr-767 with the University of North Carolina provided the services of an NDRC engineer,[d] who was stationed permanently at the Antiaircraft Artillery Board location.

In addition to assisting with the testing of all NDRC equipment at the Antiaircraft Artillery Board, and assisting the Board itself whenever asked to do so, the work involved the design and construction of a considerable number of instruments and pieces of test equipment, many of which have been built in the University of North Carolina shops. In this way it was possible to obtain quickly such items as would otherwise hold up the testing of important antiaircraft equipment. At times extra personnel was obtained from the University of North Carolina.

Apparatus constructed at the University of North Carolina included the following major items:

1. A film-type slide rule suggested by Section 7.1 for use in speeding up the necessary trigonometric computations involved in director testing.[40]
2. Apparatus for use with the precision kinetheodolites.
3. Stereo photographic apparatus for assessment of fire in which tracer is used.

2.7.6 Automatic Weapons Assessment

The development of dynamic testing equipment for directors gave NDRC a satisfactory means of comparing fire-control apparatus for guns as distinguished from automatic weapons. These dynamic director testers could be, and were, modified to make dynamic tests of certain types of automatic weapons fire-control equipment, but they were not suitable for fire-control apparatus which involves the adjustment of fire by observation of the tracer. Considerable thought was given to the possibility of constructing a device similar to that made by Section 7.2 at the University of Texas for airborne gunnery assessment (see Volume 3). In this device an optical target is presented to the gunner by means of an optical system mounted on a car which moves up and down a crescent shape track. The track itself can be moved about a vertical axis. The automatic weapons problem is complicated, first, by the fact that equipment being tested is several times as large as that used for airborne gunnery, and, in addition, it should be appreciably more accurate; furthermore, the fire control of automatic weapons depends upon tracers and it would be necessary to simulate at least part of the tracer path in order to obtain a true assessment. A mechanism was proposed which appeared to meet all the requirements and to be at least worthy of further study.

The Armour Research Foundation was consulted and undertook a preliminary study to determine:

1. Whether or not in their opinion the proposed scheme actually did meet the requirements.
2. Whether the above scheme had elements which were impossible or impracticable mechanically.
3. The time and effort required to construct the mechanism.
4. Whether any alternatives which might be better or cheaper or more practical could be found.

The answer to item 1 turned out to be that the scheme would meet the requirements, and to item 2 that the elements were possible and probably practicable but some of them quite difficult to accomplish.

However, the project turned out to be so large (several hundred thousand dollars and three years' time would be required) that it

[d] Paul Mooney.

seemed unwise to undertake it at the time, and the Armour group was unable to find any modification which would meet the requirements and yet would take substantially less time and effort.

The problem of testing AW fire control is one, however, which should be given considerable thought in the future. The present method of firing is of course unsatisfactory because of the difficulty of coordinating the flight of a test target with the activities on the ground, the dependence of such tests on weather conditions, the expenditure of ammunition and fuel and inability of a target towed by a plane to reproduce courses or to simulate high-speed or evasive flight.

Chapter 3

SERVOMECHANISMS

3.1 ORIENTATION

THE IMPORTANCE of servomechanisms in fire-control devices was recognized at the founding of the Fire-Control Subcommittee (Section D-2, later to become Division 7) of the National Defense Research Committee. In fact, the preliminary agenda for the initial meeting of the committee (dated August 1, 1940) stressed as one line of activity for the group, "A basic program of development of servomechanisms." Hence it is not strange to find that the first and second of the Reports to the Services issued by the Fire-Control Subcommittee were devoted to this subject,[1,2] resulting from the initial contract of Section D-2.

At the time of writing of this Summary Technical Report there has already appeared a number of treatises on servomechanisms which represent the state of the art at this date (1946). Several documents have been issued by Division 7,[2,3,4] others have been published by commercial houses.

One of the latter is an excellent and concise treatise[5] written at the request of Division 7 and the Applied Mathematics Panel of NDRC. Because of the timeliness of this book and its position relative to the NDRC activities, little need be recounted here as to the scope, importance, and present state of the servomechanism art. Indeed, a servomechanism will not even be defined here, since the Foreword and the first three chapters of the book[5] are devoted to this very task.

Certainly these few words summarizing the state of the servo art would be incomplete without a reference to the long history of automatic controls leading to the development of the modern servomechanism. A recent paper[6] provides an excellent bibliography on controls and leads one to the present from the centrifugal governor invented by Huygens as a possible means of regulating a clock and which was subsequently adapted by Watt to the speed control of the steam engine. In this paper are reviewed the various theories of stabilization of control devices starting with the rather tedious classical treatment to the recent simplified graphical methods of the communication engineer.

The following sections summarizing the servo activity of Section 7.3 will therefore merely restate briefly specific section projects within this field.

3.2 GENERAL ASPECTS OF SERVOMECHANISMS

Of the sixteen contracts which were the responsibility of Section 7.3, six were concerned directly and five indirectly with servomechanisms. Only one project, the first contract negotiated by Section D-2, dealt with general aspects of the problem; the remainder dealt with specific Service problems or fields which were felt to be somewhat neglected.

In 1940 the position of the servomechanism with respect to the system of which it is a component was not universally appreciated. This was particularly so with regard to military applications where secrecy considerations had resulted in piecemeal engineering of the separate components of a complete system. Recognizing these facts, the Fire-Control Subcommittee sought to undertake a substantial basic research program. Contract NDCrc-163 (Project 1) was accordingly initiated with the Massachusetts Institute of Technology and five problems of a basic nature were outlined for solution.[1]

During the course of this program, however, a part of the contractor's personnel was urgently needed for immediate Service problems under other agencies and this more ambitious program was curtailed.[7]

Two reports were issued, however, under this contract, although one, a paper on servomechanisms behavior and design,[2] was merely printed and distributed under NDRC auspices. The paper was initially prepared in response to an invitation from the American Society of Mechanical Engineers, but because of its timeliness and pertinence to the National Defense Program it was thought that the material should be distributed as a classified publication. The other report pertaining to a relay controller[8] reported the development of a particular device required at that time by the Army, namely, an automatic contact-type controller which provided a means of replacing the manual matching operation necessary for the proper operation of the fuze setter M8.

3.3 HYDRAULIC BOOSTER SYSTEMS FOR SMALL GUNS

At the request of the Armament Laboratory of the Air Corps (Service Directive AC-28), Project 15 was undertaken under contract OEMsr-18 (later OEMsr-173) was the United Shoe Machinery Corp. to develop power boosters to aid gunners in maneuvering machine guns against the aerodynamic forces on the guns. In particular, boosters were desired for the flexible single .50-caliber machine guns mounted in the rear center of the A20B light bomber, and for the twin .50-caliber machine gun mount for the tail of the B17E heavy bomber.

The mount developed, delivered for installation in the A20B plane, used a velocity-type control. Two gun mounts complete with servos, delivered for tests in B17E bombers, presented two solutions to that problem.[9] One of these mounts had a velocity-type control, the other a displacement servo wherein a certain angular movement of handgrips resulted in the proportional angular movement of the gun. Also, this latter mount had two azimuth axes, one for each of the two .50-caliber guns, in order to eliminate torques due to recoil. The other mount had a single azimuth axis, the servo having to withstand the firing torques.

The work under contract OEMsr-18 was continued under OEMsr-173, and sought to improve the first A20B hydraulic gun control unit developed and to improve also the B17E gun control units, including the addition of a pantograph mount for an optical sight and aided tracking control.[10] With the completion of these units and their transfer to the Army Air Corps, further work at the United Shoe Machinery Corp. on hydraulic boosters for machine guns mounted in aircraft was taken over by direct contract with the Army Air Corps in accordance with an agreement between Section D-2 and the Armament Laboratory, Wright Field.

While the project under contracts OEMsr-18 and 173 called for specific design of boosters and hydraulic servos for specific gun mounts, a project was initiated with the same contractor under OEMsr-19 (Project 16) for a broader research and development program of servos for aircraft gun mounts. It was accordingly recommended that thought be given to the design and construction of means for attaining aided tracking with hydraulic servos. Consideration was also given to the development of improved types of variable displacement pumps and also to develop hydraulic servos suitable for operation from gyroscopes to afford lead computation for the firing of guns. Only the aided tracking portion of the development successfully matured.[11] Further contracts for producing a more compact design of booster for application to particular Air Corps gun mounts were placed with the United Shoe Machinery Corporation directly by the Air Corps.

3.4 SERVOS FOR MEDIUM CALIBER GUNS

In May 1942, contract OEMsr-686 was negotiated with the Westinghouse Electric and Manufacturing Co. (Project 46) which represented the culmination of development work under projects with two other contractors, OEMsr-964 (Project 27) with the Barber-Colman Co., and OEMsr-522 (Project 35) with the Massachusetts Institute of Technology. The Westinghouse contract called for the production design of servos for medium caliber guns.[12]

3.4.1 Research at the Barber-Colman Company

For some time prior to the setting up of a formal contract with the Barber-Colman Co., that organization had carried on at its own expense, but with Division 7 engineering assistance, the development of clutch-type servos.[13] In April 1943, a contract was negotiated for the furtherance of this effort under the heading

"Development of Stabilizing Means for Servos." Although under a broad program, the project was rendered concrete by designing the stabilizing means for a clutch-type servo; the stabilizer, however, was of such a nature that it could be used with any other servo which has an output torque approximately proportional to an input of voltage or current. This stabilizing means was also an example of "output control," i.e., the stabilizing signal was made up of one or more derivatives of the position of the output shaft of the servo, in contrast to stabilizing means which are derivatives of the error or of the input.

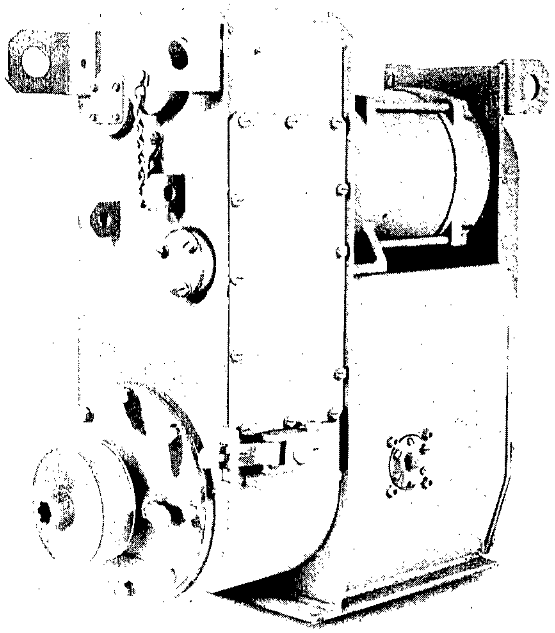


FIGURE 1. Oil Gear M3B1, right side view, showing electric drive motor (in top recess), boresighting clutch in engaged position and output coupling (lower left).

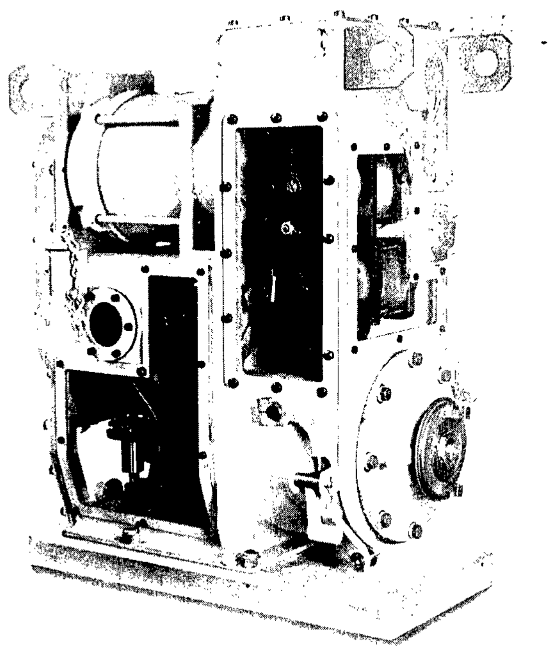


FIGURE 2. Oil Gear M3B1, left side view with covers removed and transparent windows substituted to show interior. Slewing clutch lock and lever and output coupling are at lower right. This gear may be adapted for use as either azimuth or elevation unit by adjusting gears, cams, pins, etc., in compartment to right of electric drive motor.

3.4.2 Research at the Massachusetts Institute of Technology

Concurrently with this program a project was started at the Massachusetts Institute of Technology under Ordnance Directive OD-53. The project had two objectives. The first objective was to study the Army remote control systems M1 for the 37-mm gun mount and the remote control system M5 for the 40-mm gun mount, and, if possible, increase the output torque at low output speeds and eliminate the large velocity lag of the system. The second objective was to design a new hydraulic remote control system, the system to have considerably more output torque than the M1 and M5 systems, to have essentially no velocity lag, and to be interchangeable with the M1 and M5 systems. Both objectives were met at about the same time. In view of the superiority of the completely new servo system, and its prompt adoption by the Army as Oil Gear M3B1 (Figures 1 and 2),[14] no appreciable use was made by the Army of the indicated means of improving the old M1 and M5 systems.

3.4.3 Prototype Design at Westinghouse

The hydraulic servo developed at the Massachusetts Institute of Technology underwent exhaustive tests at the Aberdeen Proving Ground and the Antiaircraft Artillery Board, and performed well. A project was accordingly set up at the Westinghouse Electric and Manufacturing Co.[12] for a production design of these servos. In view of the urgent need for the improved servo the Ordnance Department took over direction of the production design and procurement of these servos for several batteries of guns, with a view of standardizing the design. A pilot order of 100 units was placed with Westinghouse by the Ordnance Department, and on the basis of tests and field trials the Ordnance Department changed its production of servos for 37- and 40-mm gun mounts to this M3B1 design. The systems employing the new servo were termed the remote control system M9 for the 37-mm gun mount and the remote control system M10 for the 40-mm gun mount.

3.4.4 Series of Reports Representative of the State of Servo Art (Fall 1943)

At the conclusion of this work a series of Reports to the Services[4,14,15,16,17] were issued which represented at that date (December 1943) a rather comprehensive coverage of the linear servo or automatic control theory and its use in the design of automatic control devices for particular application. In making this material available, the division sought to bring together papers covering work sponsored at the Servomechanisms Laboratory at the Massachusetts Institute of Technology by the three different sources, namely, contracts with the Army Ordnance Department, contracts with Section D-2 of NDRC, and independent work of the laboratory. Acting as a distributing agent with the approval of the sponsors of the work, Division 7 sought to disseminate documents which, taken together, formed a treatise on the current (fall 1943) status of the servo art, in particular small servos for application to small- and medium-caliber guns.

3.5 SPEED REGULATOR FOR MOTORS AND MOTOR GENERATORS

Completing this listing of servomechanisms developed under Section 7.3 auspices is a speed regulator for an airborne motor generator set (Figure 3) developed to satisfy a requirement

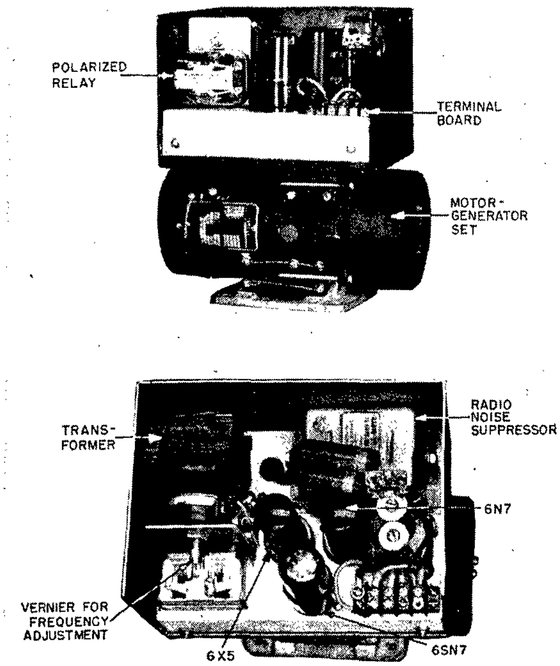


FIGURE 3. Electronic controlled motor-generator set—relay type. Phase sensitive electronic circuit for regulating speed of motor-generator, contained within box on top of M-G set, is shown open.

for precise gyro speeds in certain gyroscopic instruments which will be discussed below. The requirement was for electric power of frequency regulated within a fraction of 1 per cent. The section sought to develop a speed regulator for a standard Army-Navy airborne motor-generator set powered by the usual 28-volt d-c aircraft supply. Additionally, accurate regulation of the a-c frequency of the generator was required even with a supply voltage variation from 24 to 32 volts. The project was undertaken by the Leeds & Northrup Co. under contract OEMsr-1292 (Project 81) to develop a speed control utilizing a servo loop which corrected the d-c motor speed in accordance with deviation error of the generated frequency from 400 cycles per second. The project resulted in the development of a device which maintained a frequency within a few thousandths of 1 per cent at constant d-c voltage, and within 1 per cent with supply voltage variation between 19 and 36 volts. Several motor-generator sets equipped with this speed control circuit were turned over to the Navy for use in connection with their program on the manufacture and tests of the bombsight Mark 23.[19]

3.6 SEACOAST DATA TRANSMISSION SYSTEMS

3.6.1 Development at the Bell Telephone Laboratories

In March 1941, the Coast Artillery Board initiated Project 1207 for the purpose of obtaining a satisfactory solution to the problem of continuously transmitting data from base end stations to gun data computers. Contract

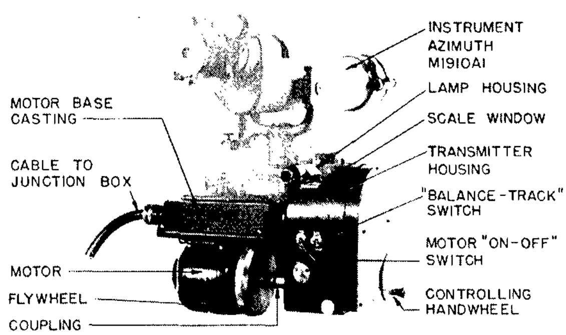


FIGURE 4. Seacoast data transmission transmitter, azimuth, M7 showing transmission end of resistance bridge telemeter affixed to standard telescope receiver.

OEMsr-177 (Project 20) was negotiated with the Bell Telephone Laboratories for this purpose, and two alternate systems were developed: a voltage divider type[20] and a permutation code type.[21] Both systems gave excellent performance as regards both accuracy and reliability. However, because of the relative simplicity of the former type over the latter, the voltage divider system was standardized by the Coast Artillery Board and the Ordnance Department.

The system is basically a high-resistance d-c Wheatstone bridge, consisting of two potentiometers, line-balancing rheostats, a power source, and a meter indicating unbalance. The arrangement is unique in that the potentiometers have two brushes, and the troublesome discontinuity occurring at the ends of the potentiometers is avoided by always using the center portion of the potentiometers.

3.6.2 Pilot Model Development at Leeds and Northrup Company

Subsequently, contract OEMsr-404 (Project 34) was negotiated with the Leeds and Northrup Co. to develop a pilot model of the voltage divider system. The transmitter assemblies, utilizing "aided tracking" for tracking the target were designed for field mounting on azimuth instrument M1910A1 (Figures 4 and 5) and depression position finder M1, and the receivers for field mounting on seacoast gun data computer M1.[22]

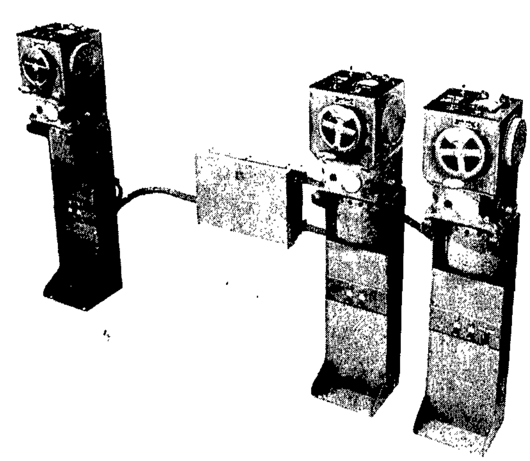


FIGURE 5. Seacoast data transmission M12 and M13 receivers.

Pilot models of two transmitters and two receivers were built and in September 1942 installed at Fort Story, Virginia. After tests on these prototypes by the Coast Artillery Board, production orders were placed by the Ordnance Department. Toward the end of the development contract, the potentiometers were modified for transmissions from the SCR-296 seacoast gunlaying radar.

3.6.3 Mechanism to Measure the Smoothness of Control of Aircraft Turrets

The Bureau of Aeronautics requested the development of an aircraft turret smoothness tester. This project was placed with the Waugh Equipment Co. (Contract OEMsr-1185, Project 75).

The Bureau of Aeronautics required an instrument capable of testing the smoothness of aircraft turrets to place acceptance of production turrets on a quantitative basis. A primary requirement was to develop an instrument which did not involve making shaft connections to the power equipment in the turret, so as to facilitate installation of the test equipment.

An attempt to use commercially available instruments failed because of the very low frequencies which had to be measured. After trying several expedients a capacitance-type pickup was designed which responded to very small accelerations and low frequencies, and which was deemed satisfactory for the purpose by the Navy.[23]

Chapter 4

PNEUMATIC CONTROLS

4.1 GYROSCOPIC LEAD-COMPUTING SIGHTS

In August 1941, Section D-2 distributed a report to the Services[1] on the fundamental dynamics of the gyroscopic lead-computing sight.[a] This paper was the section's reaction to the lack of adequate mathematical material on the subject. This study was supplemented in August 1942 by a second report[2] which sought to question how a lead-computing sight should be designed to give the highest possible accuracy in its computation of the leads for a straight-line target.

These general studies promptly led to section interest in basic improvements of the mechanization of gyroscopic lead-computing sights. Two broad objectives were effected: (1) mechanical refinement of the data-computing elements of the instrument, and (2) operational improvement by mechanical modification of the data presentation means utilized by the sight. The former is self evident since improving the computation should give better results. The latter involves a psychological problem due to the nature of the instrument resulting from the fact that a lead-computing sight is a "disturbed" sight.[b]

The first objective involved two types of data: the time of flight of the projectile, or range, and the angular rate of the target. Range depends upon some target observing means, radar range finders, optical range finders, or just sheer guess, and can be easily introduced into a computing mechanism by a hand crank or automatic means. This data can be as accurate as desired. Angular rate is obtained by a suitable mechanism which measures the angular rate of the target relative to the gun. In order not to measure the angular motion of the platform on which the gun is mounted it is generally desirable to employ a gyro for the angular rate measurement. One specific task that Section 7.3 embarked upon was to improve the gyroscopic rate of turn indicator.

The second objective was the more elusive in that it was desired to present to the gunner a line of sight offset by the correct lead angle during steady-state tracking, but to decrease the time required for the transient solution. To decrease the transient solution time constant to less than approximately 1.3 times the time of flight setting, and to erase false lead angles due to slewing on to the target required some form of chicanery.

4.1.1 Pneumatic Gyroscopic Lead-Computing Sight

In exploring methods of mechanization attention was focused on pneumatic controls, which had long been used in industrial instrumentation but had not been exploited in the fire-control field. The usual method of measurement of angular rates by gyroscopic rate of turn indicators involves constraining a dynamical system, and measuring the forces of constraint by springs and measuring the force by noting the deflection. In order to get a substantial response the spring must stretch considerably to actuate a pointer or some other element. In thus stretching the spring, the gyroscope must move, i.e., be imperfectly constrained, and energy is stored in the spring.

[a] Although the material was specifically directed at gyroscopic lead-computing sights, a large part of the discussion applied to any sighting mechanism which, the gunner directly controlling the gun, sets up between the gun and the optical line of sight a lead angle which is the product of the time of flight by an angular velocity.

[b] In a lead-computing sight the gunner exercises direct control over the position of the gun and not the line of sight. The gunner, by moving his gun, causes motion of the line of sight by means of the computing mechanism and, eventually, tracks the target with the gun displaced in accordance with the proper lead angle. Since the gunner has only indirect control over the line of sight, a confusing psychological situation may obtain, for a given motion of the gun in general results in a different motion of the line of sight. The gunner's hands, on the gun, attempt to produce a certain result, but his eyes see something else happen. For this reason a sight of this sort is called a "disturbed" sight.

In an effort to constrain stiffly a gyroscope and thus more nearly approach perfect constraint, it was thought that a gyroscope could be constrained by forces caused by fluid pressures which would be varied by *slight* motions of the gyroscopic element, and the fluid pressures would in turn be used to actuate an indicator.

4.1.2 Development at the McMath-Hulbert Observatory

Accordingly, contract OEMsr-504 was set up at the McMath-Hulbert Observatory (Project 40) for the purpose of developing a gyroscopic lead-computing sight embodying pneumatic elements, with primary consideration given to the development of a pneumatically constrained gyroscopic rate of turn indicator. This purpose was achieved to the extent that a rate of turn indicator was developed which would measure angular rates from several hundred to a tenth of a mil per second with a full-scale pressure signal of a few pounds per square inch. Furthermore, the gyroscope was itself constrained to within a fraction of a mil angular motion.[3] However, after the development of the pneumatically constrained gyroscope the exigencies of World War II were such that the contractor's efforts were redirected to an urgent program for the development of an angular rate bombsight. Progress Reports 1 to 11 appended to the McMath-Hulbert Observatory report[3] give an account of the lead-computing sight activity.

4.1.3 Development at the Eastman Kodak Company[c]

With the shift of emphasis at the observatory to a program of development of a bombsight, the Eastman Kodak Co. was asked to set up a program under their already active contract OEMsr-56 (Project 17) for the development of a pneumatic gyroscopic lead-computing sight.[4] This was to take the form of a modification of the standard Navy gunsight Mark 15 which was being produced at the Kodak Co., and designated as the Mark 15-P.

This problem was also to be interrupted by the urgent bombsight program mentioned above when the Eastman Kodak Co. was asked to design a production version of the bombsight developed at the McMath-Hulbert Observatory. Before completion of the gunsight problem, however, progress pointed clearly to the probability of a successful conclusion. However, termination of hostilities came before development could be again started and this development must finally be listed under a heading of unfinished business at the end of the Eastman contract. It is felt that this device, if finished, will provide a gunsight embodying several outstanding features which are recounted below after reviewing the problem in a little more detail.

4.1.4 The Navy Gyroscopic Lead-Computing Gunsight Mark 15

As stated above, the Eastman Kodak Co. was asked to modify a standard Navy gunsight Mark 15. This gunsight is a typical gyroscopic lead-computing sight based on the assumption that the angle by which the gun must lead the line of sight is equal to the angular rate of the gun times the time of flight of the shell to the target. This assumption is approximate and corrections are made by the use of variable filter constants, arbitrary functions of range, and superelevation corrections.

It will be recalled that a lead-computing sight requires the use of some type of smoothing circuit in the output, but this filtering introduces a time lag[2] and a "history." That is, all the tracking which goes into slewing the sight so that it is on target is remembered by the filter circuit. This causes errors for a substantial fraction of the course, even after correct tracking has been established.

In the Mark 15 gunsight, two gyroscopes are used to measure the elevation and traverse components of angular rate. These are coupled

[c] This account omits reference to early exploratory work in connection with controlled reticles for lead-computing sights. It was hoped that persistence of a fluorescent reticle would introduce the necessary smoothing of the lead data. For complete details see reference 16 of Chapter 2.

directly to mirrors which "set in" the components of lead angle between the gun and the line of sight. Damping is introduced by viscous dashpots mounted on the gyroscopes themselves, and a spring with a variable gradient exerts a torque on the gyroscope which is inversely proportional to the time of flight setting. As a result, the time constant of the filter is proportional to the time of flight.

4.1.5 Principles of the Mark 15-P Sight[4]

Differing from the Mark 15 in several respects, the general scheme of the Mark 15-P is as follows: A pin joining two slotted cranks is used to multiply the precession torque of the gyroscope by a factor that represents the time of flight (Figure 1). This resultant torque is measured as a pneumatic pressure difference by a pressure pickup, the pressures are transmitted through a pair of variable pneumatic filters, amplified by a pair of pneumatic amplifiers, and finally made to drive mirrors which deflect the line of sight. These operations are described more fully in the report.[4]

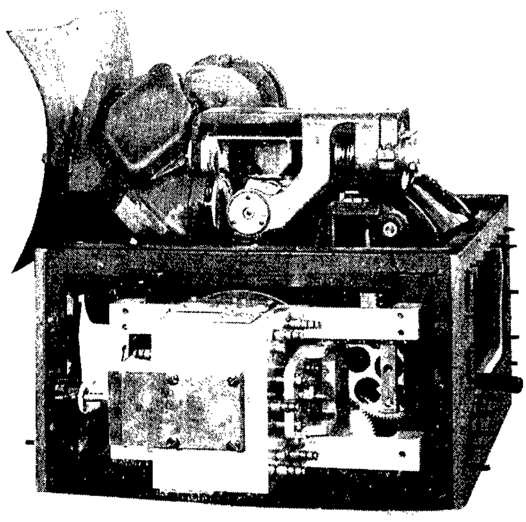


FIGURE 1. Lead-computing sight "Mark 15-P."

Standard Navy Mark 15 sight modified for pneumatic constrained gyroscope and pneumatic computing elements, covers removed, exposing added (shiny) equipment. This view shows the extreme compactness of the pneumatic computation elements. This one block, only 4⅛x4⅜x3⅛ inches, contains four pneumatic amplifiers, four variable filter elements, four pressure-operated valves, four calibrated pneumatic resistors, and a servo system.

4.1.6 Advantages of the Mark 15-P System

The primary advantages of the Mark 15-P over the Mark 15 are:

1. The pneumatic system between the gyroscopes and the mirrors makes it possible to turn the computation on or off, and to throw away the history at will. Thus this system allows the tracker (a) to swing the sight on target with fixed optics, (b) to cause the approximate lead angle to be applied rapidly to the optics, and (c) to introduce the proper time constant for that particular range.
2. Since the pneumatic system does away with direct mechanical connections between the gyroscopes and the mirrors, it eliminates vibration troubles.
3. A variable time constant makes possible a selection of the best relation between time of flight and the time constant of the filter, which yields a more accurate computation.

4.2 DEVELOPMENT OF NAVY BOMBSIGHT MARK 23

The spring of 1942 found German submarine activity against Allied shipping so great and so effective that ships were being sunk at a rate greater than they could be built. This situation called for an urgent program on countermeasures. Work was accordingly initiated with the Franklin Institute, under the auspices of Section D-2, and later Section 7.2, for development of bombsights suitable for directing depth charges against submarines from low-flying aircraft. In particular, there was developed a bombsight of the conventional "depression angle" type, which was standardized by the Navy as the bombsight Mark 20. (See Volume 3.)

4.2.1 The British Angular Rate Bombsight Mark III

Early in 1943 the British advanced the broad idea of determining the bomb release point by measurement of the angular rate of the line of sight. This was in contrast to the conventional method of using the angle of the line of sight from the horizontal as the release criterion. They mechanized this scheme in a device known as the British low-level bombsight Mark III.[5] This sight involved visual comparison of the

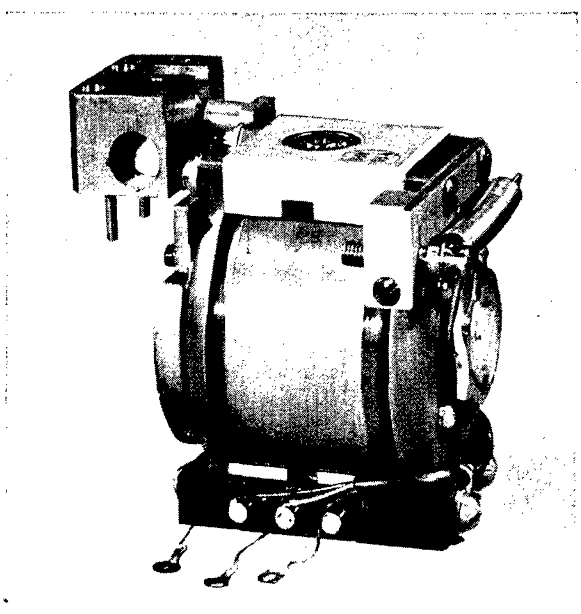


FIGURE 2. Pneumatically constrained gyroscope for Mark 23 bombsight.

The sensitive element of the Mark 23 is a 400-cycle 3ϕ electric gyro. When precessed, the gyrodynamic action tends to relieve a spring (upper right side) tensioned in accordance with a bombing problem. The gyro is captured by a pneumatic potentiometer (upper left), comprising a slotted block rigidly affixed to the housing and a cup-pair contained in the slot attached to an arm affixed to the gyro. The gyro is thus restricted to angular motions of a fraction of an angular mil, the clearance of the cups and the block. The pneumatic take-offs are the two tubes extending beneath the block.

relative angular rate of the line of sight and an illuminated moving reticle, with manual release of bombs by the bombardier at the instant of zero relative rate. Because of the necessity of judging zero rate and because of the necessary human reaction time for releasing the bombs, it was felt desirable to initiate other mechanizations of this principle. Work was therefore started at the Franklin Institute under Section 7.2 auspices to study the British principles and to develop an improved instrument. The study took the form of a mathematical evaluation of the various types of bombsights, angular depression, angular rate, and angular acceleration.[6] (See Volume 3.)

4.2.2 Development at the McMath-Hulbert Observatory

It was recognized that the new pneumatic gyroscopic angular rate indicator (Figure 2) developed for use in a lead-computing sight could serve as the basis for a bombsight wherein the target would be tracked and bombs could be released automatically. The McMath-Hulbert Observatory project was redirected for the development of a low-altitude angular rate bombsight. It was felt that not only would the pneumatic bombsight give accurate determination of the release point and afford a simple method of incorporating smoothing of the tracking data, but would also provide simplicity of design

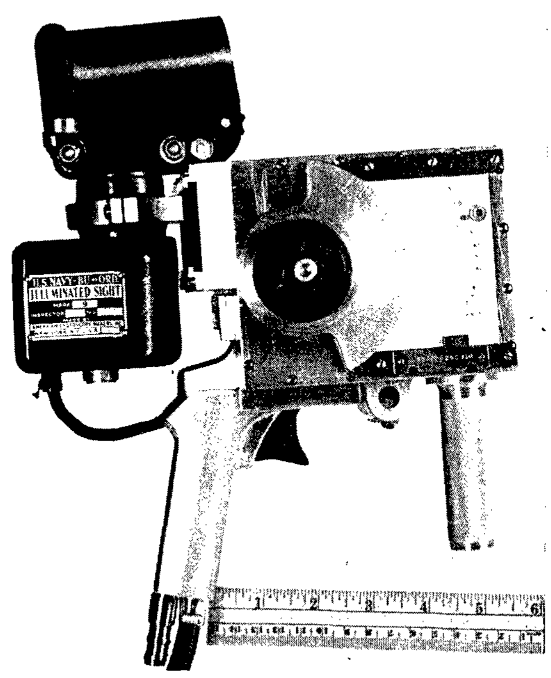


FIGURE 3. Bombsight BARB III, Model 2.

Originally contemplated for hand-held use, this device went through many stages of evolution. It finally resulted in a device mounted on trunnions but hand-tracked, which was standardized by the Navy as the bombsight Mark 23.

and conservation of weight and space. Work was accordingly started at the observatory[3] on the development of an instrument incorporating these principles (Figure 3), which was informally designated BARB III (British Angular Rate Bombsight). Parenthetically, the mathematical problems involved were intricate, since smoothing of the tracking data required the solution of certain nonlinear equations; the resources of the Applied Mathematics Panel, NDRC, were accordingly drawn upon in their solution.[7]

In June 1943, a bombsight, hurriedly built and embodying the pneumatic features, was given preliminary flight tests by the Navy at Quonset Point Naval Air Station. These tests proved especially gratifying, and the design and construction of further improved sights incorporating changes recommended by the Navy were undertaken at the McMath-Hulbert Observatory.

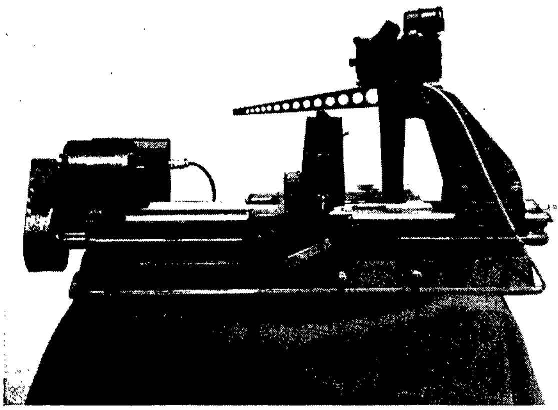


FIGURE 4. Bombsight calibration outfit Mark 2, Model O.

This device provides a miniature of the horizontal low-level bombing problem (scale 1/3,000). A Mark 23 bombsight is shown mounted for test. The lead screw carrys an adjustable roller at a constant rate (representing a target) and the lever arm resting on the roller angularly swings the bombsight as if it were being tracked. The bombing problem is preset on the bombsight and the calibration outfit. One of the scales to be set on the calibration outfit may be clearly seen on the roller carriage, and the other scale is positioned by manipulating the hand crank on the extreme right of the outfit.

Certain contributing projects were also begun. The first of these was for the development of a bombsight testing engine[3] for calibrating the finished sight. This instrument afforded a small-scale replica of the actual bombing run (scale 1/3,000), the bombsight being subjected to ideal angular motion in accordance with a preset problem, and the error in release time given as a galvanometer deflection. This instrument was later standardized by the Navy as bombsight calibration outfit Mark 2, Model O (Figure 4). The second item developed was a bombrack delay timer.[3] This afforded a simple means of recording the interval of time between the bombsight signal and the release of the bomb by the bombrack.

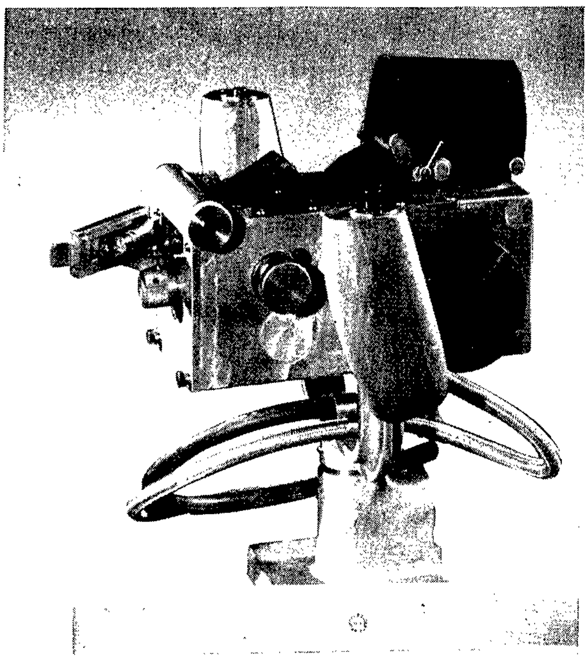


FIGURE 5. Bombsight Mark 23.

This pneumatic bombsight is hand-tracked (counterclockwise) while maintaining an illuminated collimated reticle of the optical sight on target. Electric power for the gyroscope and bomb release mechanism is supplied by the armored cable and the case is evacuated through the rubber hose. The bombing problem is preset by manipulating the two knobs on the upper left corner of the case while viewing a nomograph through the prismatic (black) lens holder on the top of the case. The lamp which illuminated the nomograph is exposed. Access plates to the pneumatic computing elements are shown in place on the top of the case (rectangular plate, abutting front edge of the black optical sight hood), and on the side (circular plate, lower right corner). The button on the top of the right-hand grip acts to arm the sight, and the one on the left-hand grip actuates the bombardier's microphone.

4.2.3 Development at the Eastman Kodak Company

The flight test results of the bombsight BARB III led Section 7.3, in anticipation of a Navy production program, to request the Eastman Kodak Co. to divert its facilities and manpower from the lead-computing sight project to a program of designing the experimental sight for production.[8]

4.2.4 Navy Production of the Mark 23 Bombsight

While the Eastman Kodak work on a production design was still underway the Navy decided to produce the pneumatic angular rate bombsight, and standardized it as the bombsight Mark 23 (Figure 5). The Eastman Kodak Co. drawings were turned over to the Navy's contractor, the American Cystoscope Co.

For the reader's convenience, the tables of contents of the McMath-Hulbert Observatory and the Eastman Kodak reports are reproduced here and serve as an indication of their contents and scope.

Final Report on the Mark 23
Low-Altitude Angular Rate Bombsight,
McMath-Hulbert Observatory [3]

Angular Rate Bombsight Mark 23
Development Dept., Eastman Kodak Co.[8]

4.3 BOMBSIGHT MARK 25

While the performance of the bombsight Mark 23 was better than that of the other low-altitude bombsights it was felt that a superior mechanization could be developed. In particular, it was felt that if the optics could be fully stabilized and not subject to the angular motions of the aircraft considerable advantage could be gained over an instrument which was not so stabilized. Also, it was felt that tracking by means of a handwheel geared to the optics with a reasonable ratio would be preferable to direct control of the line of sight; furthermore, it was considered desirable to make the connection between the handwheel and the optics one involving "aided tracking."

Work on a stabilized version of BARB was started in December 1943, interrupted during intense activity on the Mark 23 program between April and December 1944, and finally completed in January 1946.[9] The mechanization of this sight is more complex than that of the Mark 23. It is pertinent to mention that Navy tests of this bombsight showed it to be capable of solving the low-altitude bombing problem with extreme accuracy. This prompted the Bureau of Ordnance to standardize the sight as the bombsight Mark 25, Mod O, (Figure 6), and to request Division 7 to transfer all apparatus and instruments developed at the Eastman Kodak Co. in connection with this project to the Naval Ordnance Plant in Indianapolis, Indiana, for continuation of this work. A formal Navy report on flight tests of the sight is still forthcoming at the time of the writing of this report.

4.4 PNEUMATIC CONTROL ELEMENTS FOR FIRE-CONTROL APPLICATIONS

Early in 1944 it became apparent to Section 7.3 that a major contribution could be made in the field of aircraft instrumentation by exploiting the pneumatic techniques which led to the development of the bombsights Mark 23 and 25 and the lead-computing sight Mark 15-P. Accordingly a broadly defined project was set up at the Lawrance Aeronautical Corporation under contract OEMsr-1366 (Project 82).

This contract was for the development of compressible fluid controls and stabilizing means of general application for fire-control devices. In particular, four groups of projects were un-

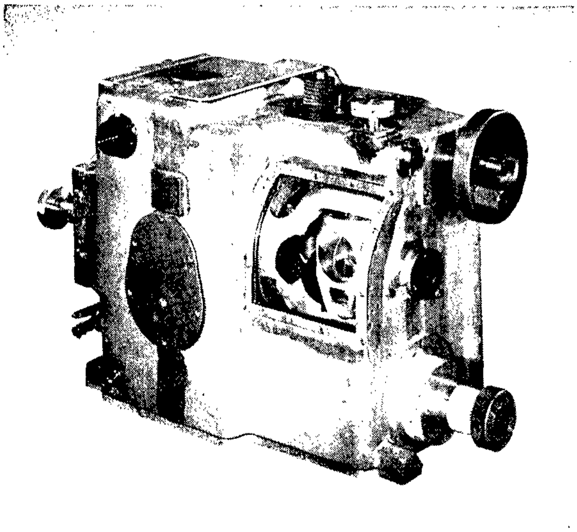


FIGURE 6. Bombsight Mark 25.

The housing serves as a rigid support for the gimbal frame of a free gyroscope and forms an airtight case about the components of the sight. The torque necessary to precess the gyro in elevation tracking is applied to the frame of the gyro by a pneumatic device actuated by the hand crank (upper right), and the bombs are automatically dropped in accordance with a problem preset by manipulation of the knobs bearing the nomograph (lower right). The bombardier views the target through the window (front, right) and an optical system carried by the gyroscope gimbal. Lateral tracking is given by the knob at the top of the case. The protective shell for a pneumatic filter is also shown on top of the case. The case is evacuated through the pet cock on the left side of the case (lower left) to supply a working pressure difference for the pneumatic mechanisms. The luminous intensity of the optical sight is controlled by the black knob on the upper left front side of the case.

dertaken, two of which were supplemental to work being done on other contracts.

4.4.1 Vacuum Regulator for Mark 23 Bombsight

In the course of the development of the Mark 23 bombsight, a standard vacuum regulator was adapted to control the vacuum supply to the bombsight head. Unfortunately this vacuum control was of the "dump" type which allowed atmospheric air to flow into the valve through a dump port when the vacuum exceeded a certain amount. Certain troubles were experienced in connection with the Mark 23 bombsight and it was felt that these troubles would be alleviated if a throttle-type regulator were used. Since the production sights were well under way and since space requirements were such that standard throttle-type regulators could not be used, a throttle-type regulator of unusual design was developed of identical space factor with the standard regulator.[10] Several prototypes were made for extended tests and samples made available to both the Army and Navy.

4.4.2 Elements for Torpedo Control

In collaboration with Division 6, Section 7.3 extended its efforts to the field of torpedo controls under contract OEMsr-1144 (Project 69) with the Foxboro Co.[11] The project was set up to develop improved steering mechanisms for torpedoes, both in azimuth and in depth. However, the azimuth steering work was carried on at the laboratories of the Division 6 contractor. The aim of the depth control development was predicated upon an indication that the standard depth control was difficult to manufacture and marginal in performance, and satisfactory depth keeping was obtained only when the unit was very carefully made and was adjusted by one of a few expert technicians. Thus this project sought to develop a depth control which would (1) be easier to manufacture, (2) give better performance both as regards recovery from launching and during the run, (3) be less subject to fore and aft accelerations, (4) be preferably lighter in weight, and (5) be more rugged than the present control. Test runs of this control were made in torpedoes at the Newport Torpedo Station, but none of the controls performed satisfactorily. In the time available and consistent with the other war commitments of the company, it was not possible for the contractor to complete the development of an improved depth unit. Much progress was made however, and inasmuch as the general features of the design appear sound, it is expected that the Navy will continue this work.

In furtherance of the work at the Foxboro Co. a project was set up under the Lawrance Aeronautical Corporation contract for the improvement and development of a depth engine for torpedoes.[12] This depth engine was not a sufficient improvement over the present standard device to warrant a change, and the work accordingly was not extended beyond the developmental stage.

A third torpedo component developed at the Lawrance Aeronautical Corporation was a 500-psi torpedo regulating valve, which was the pressure reducing valve for the 3,000-lb air supply. The objective was to design a regulator valve for torpedoes for easy manufacture and for stability of operation without expert adjustment. The regulator from a Mark 15 torpedo was modified to this end and laboratory tests indicate the design to be satisfactory. The design and test information and a model were turned over to the Navy for use in guiding the subsequent design for torpedo pressure regulators.[13]

4.4.3 Stabilization of Aerial Cameras

In conjunction with a Division 16 program for the development of cameras and camera stabilizers, work was done under the Lawrance contract on the stabilization of cameras by pneumatic means. Initial work dealt with a camera stabilizer suitable for reconnaissance photography. A camera stabilizer for mapping application was undertaken after the exploratory phase was completed. The result was a laboratory model which provided excellent stabilization of the camera against motion of the platform (simulated aircraft angular motion) and which provided for maintaining the camera axis at the effective vertical within close limits.[14]

This equipment was demonstrated to the representatives of the Photographic Laboratory of Wright Field under whose auspices this project was initiated (AC-76), and the Army has placed a contract calling for further work with the engineers responsible for this development.

4.4.4 Pilots Universal Sighting System [PUSS]

In collaboration with Section 7.2, work was done under the Lawrance Aeronautical Corporation contract on pneumatic components for a pneumatic version of PUSS.[15] (See Volume 3.) Units developed under the contract for this purpose were (1) an absolute pressure regulator, reference pressure regulator, and relief valve, (2) twin variable pneumatic resistors, (3) a servo operator for use on the PUSS gyro, (4) a solenoid valve of special design, and (5) vane motors for the optics of PUSS.[15] These components with complete design and test information were turned over to the Franklin Institute for use in carrying the PUSS project to completion under a direct Navy contract (Navy contract NOrd-9644).

Also in connection with the PUSS project, work was done under contract OEMsr-56 at the Eastman Kodak Co. (Project 17) to design and construct a pneumatically constrained gyro. A model gyro was turned over to the Franklin Institute for incorporation in the system after laboratory tests indicated it was satisfactory for the application.[16]

4.4.5 Other Pneumatic Circuit Components

Four miscellaneous items were also made under the Lawrance contract, two of which were exploratory models of devices which were redesigned as PUSS components: a pressure amplifier,[17] and a pneumatic resistor.[18] The other two items developed were under an informal request by the Bureau of Ordnance for a dive angle indicator embodying the cup-constraining features used in connection with the Mark 23 and the torpedo developments.[42] Several dive angle indicators were assembled and turned over to the Bureau of Ordnance for use in its development programs, particularly with regard to toss bombing.

At the Eastman Kodak Co. a pneumatic phase-controlled speed-regulating device for air gyros[19] was developed for the purpose of adapting this unit to air-driven rate gyros in applications when precise gyro speeds are required. This speed control device employed a vibrating reed as the frequency standard, and gave excellent speed regulation.

4.5 THEORETICAL ASPECTS AND DESIGN CRITERIA

During the course of development of the various pneumatic instruments discussed above, Section 7.3 turned to the basic problem of their theory of operation. In particular, a problem was initiated under the Lawrance Aeronautical contract to study the components of the systems developed and to compile the design criteria which existed only on odd charts and in the notebooks of the engineers working on the projects.

4.5.1 Literature on Pneumatic Instrumentation

Pneumatic instruments are common in industry, where pressure, fluid flow, and similar physical quantities must be controlled. The literature covering the design and theory of these instruments is quite complete.[20,21] Certain aircraft instruments (rate of climb indicators, airspeed indicators, altimeters, etc.) are also common pneumatic devices. These have been analyzed and make up a considerable body of literature.[22-27] It may be of interest to recall that many examples of both these classes are to be found in the patent literature. Seldom, however, does there appear a reference in this field to means of accomplishing a given task pneumatically in analogy with the usual way of performing the job in other fields, say, electrically or mechanically[28] although the analogs are old and well known.[29-31]

Indeed, one must turn to the field of acoustics for an introduction to a comprehensive treatment of pneumatic theory. The modern treatises,[32,33,34] firmly rooted in the classic expositions of Rayleigh[30] and Lamb,[31] supply valuable source material for a mathematical analysis of pneumatic instrumentation phenomenon. Here we find the notions of fluid pressures as voltage, fluid flow as current, the capillary as a resistance element which serve as a basis of analysis of systems using these quantities and/or elements as components.

4.5.2 Mathematical Studies

Lacking handbooks and similar guides which give in tabular form particular values for these various quantities as a function of dimensions and ambient conditions, Section 7.3 was obliged to draw up design charts for this purpose. This required recourse to the field of acoustics and the works referred to above. Subsequently these charts were expanded and a report,[35] issued under the Lawrance contract, tabulated these data in a form similar to Massa's acoustic design charts.[36] Mathematical work of an advanced nature on particular pneumatic components was begun during the life of the contract, but was never concluded because of the termination of World War II.[37,38]

4.5.3 Comparison Study of Gyroscopes and Gyroscope Substitutes

During World War II two gyroscope substitutes were proposed. One was a "fluid gyro"[39] and another made use of a vibrating reed.[40] The fluid gyro was proposed by the Columbia University Division of War Research at the U. S. Navy Underwater Sound Laboratory, New London, Connecticut. This was fashioned as a case with a vaned passage of axial symmetry through which a fluid current is directed. The case being subjected to a torque about the axis imparts a moment of momentum to the fluid, and the angular momentum of the fluid is subsequently used as an indication of the angular velocity given the case with respect to a body pendulus with respect to the axis.

The vibrating reed made use of the fact that a rod anchored at one end and maintained in a state of vibration about its anchorage tends to maintain its vibration in a frame which is fixed in space.

A series of studies were carried out under contract OEMsr-268 with the Barber-Colman Co. to compare these two devices with an ordinary gyroscope of typical parameters and the results form appendices to the final report under that project.[41] The conclusions were summarized as follows:

1. When stabilization about one axis only is required, it is difficult to state which instrument will have the more accurate dynamic response. For sinusoidal forcing accelerations the standard gyro proved to be considerably more accurate. For the other types of motion

considered, it is impossible to say in general that either will be more accurate than the other.

2. When stabilization about two axes is required, two fluid gyros should prove more accurate than a single standard gyro for all types of forcing accelerations considered.

3. The fluid gyro should have considerably smaller static friction error than the standard gyro.

Chapter 5

MATHEMATICAL ANALYSIS OF FIRE-CONTROL PROBLEMS

5.1 INTRODUCTION

Four groups of projects to be summarized below are concerned with contributions of the Mathematical Analysis Section, Section 7.5, to the fire-control research program of NDRC. Several of these projects were begun under Section D-2 auspices and came to fruition before the reorganization of NDRC. Other projects were, for reasons of administration expediency, transferred to the Applied Mathematics Panel at the time of its organization.

Because the Chief of Section D-2 and of Section 7.5 became the Chief of the Applied Mathematics Panel, and because the Division 7 activities in mathematical analysis, as a special case, were incorporated in the more general applied mathematics picture of the panel, the present account of mathematical analysis of fire-control problems must necessarily be partial. For a more integrated and comprehensive account, the reader should consult the Summary Technical Report of the Applied Mathematics Panel.

Ten projects of Division 7 were in the province of mathematical analysis. Two dealt with statistical prediction, three with mathematical aids for computation, two may be designated as miscellaneous, and three were transferred to the Applied Mathematics Panel. These will be taken up in the order given.

5.2 STATISTICAL THEORY OF PREDICTION

It is essential, in the design of any antiaircraft or other director, to have a device which calculates some estimate of the future coordinates of the target, basing this calculation on a knowledge of the present and the past values of the coordinates and the derivatives thereof. Two main types of methods are of particular significance for the calculating of future values. One, which is the method used in the past and which will be called here the *geometrical method,* rests upon certain assumptions as to the geometrical character of the path of the target (say the assumption that it moves on a straight line), and the future position which would then result is computed on the basis of this assumption.

The actual computation of future or predicted position is rather simple. The method is inherently capable of perfect precision, the actual practical accuracy being limited by the accuracy (which may in some cases be excellent, and may in others be low) with which one knows the parameters (say present position and velocities) which determine the geometrical path in question; and by the accuracy (which may again be high or very low) with which the actual flight path of the target conforms to the assumption made as to its geometric character.

During World War II various generalizations of this basic geometrical method were proposed. These generalizations involving some type of extrapolation to future value are based upon present values and some range of past values. In this extrapolation procedure the prediction is usually expressed as the sum of terms of decreasing importance; and one uses, in the actual mechanized solution, as many terms as prove practicable. The calculation itself therefore usually involves approximation.

This extrapolation procedure is merely a generalization of the older and more familiar form of the geometrical method; for any extrapolation formula adopted constitutes an assumption as to the geometrical character of the path. The extrapolation method at first sight appears to differ in that it makes explicit use of past data as well as present data. But this distinction is an illusion, for the geometrical method, even in the simple case of straight-line motion, necessarily uses past data to compute velocities. Indeed, because of the errors inherent in target motion, observing of target motion, and computing, it is necessary to use quite a range of values in order to compute a usefully approximate value for an "instantaneous" velocity. The extrapolation method recognizes this situation explicitly, and uses a range of past data to help compute the geometrical parameters of the target path.

5.2.1 Statistical Method of Prediction (Projects 6 and 29)

Under contract NDCrc-83 (Project 6) a mathematical theory of a second and more general type of a prediction, which may be termed the *statistical method,* was worked out.[a] The statistical method, as a preliminary step, utilizes the older geometrical method to make an approximation to the prediction; but by comparing the actual present values with the first-order predictions made in the past and culminating at the present, one has available as present data the amounts of $f(t)$ by which the simple geometric or extrapolation method fails. This deviation $f(t)$ is assumed to have two additive components, the first of which, $f_1(t)$ (the "signal"), is principally due to the fact that the actual motion of the target does not conform to the assumptions made in the simple prediction; and the second of which, $f_2(t)$ (the "noise"), is due to errors of various sorts.

Wiener worked out mathematical methods which statistically examine the past behavior of the signal $f_1(t)$ and produce, at the present time t, the "best possible" guess as to what value this signal $f_1(t)$ will have at a future time. The phrase "best possible" has a definite meaning in the least square sense. The theory also included a determination of the characteristics of electrical networks which are capable of carrying out the necessary operations. It is necessary (and possible) to design the networks in a general way, leaving certain important characteristics adjustable. To set the network so as to do the best job of statistical predicting of the future for a given target, one sets the adjustable characteristics of the network in accordance with certain statistical parameters which characterize the past motion, which describe, so to speak, the "habits" of the motion. These statistical parameters are themselves determined by the machine; and the predicting network may be so flexibly adjustable that it is able to handle motions of widely differing statistical characteristics.

The preceding paragraph refers only to the signal component $f_1(t)$. The theory also provided means for the best possible filtering of the total quantity $f(t)$ into its signal component $f_1(t)$ and its noise component $f_2(t)$.

The successful performance of a simplified laboratory model of a statistical predicting network designed in accordance with the theory was demonstrated. In order to carry out such a demonstration, it was necessary to have a suitable type of input signal to predict. There was constructed, therefore, a device in which one attempted to track a sinusoidally moving spot of light by means of a second spot over which the operator had a loosely coupled control through a somewhat complicated mechanism involving large inertia, spring coupling, damping, etc. The "error" between the tracking spot and the tracked spot is then taken as the statistical input to the predicting circuit.

The circuit demonstrated its capacity to predict on such a signal over an interval of as long as 2 seconds. For a 1-second prediction the correspondence between the predicted value and the ultimately realized value was astonishingly close.

Whether or not this interesting procedure has important application to long-range antiaircraft fire control depends on the following considerations. Taking only one coordinate of the target plane, the input to the predictor contains (1) "flight errors" corresponding to actual movements of the airplane due to bumpy air or to the pilot himself, (2) long-period tracking errors, (3) short-period tracking errors, and (4) "noise" or other small errors due to miscellaneous causes. The question is: Are these various errors sufficiently separated in frequency so that Wiener's network can predict the flight errors and at the same time filter

[a] By Norbert Wiener of the Massachusetts Institute of Technology.

out (or at least not predict) the long-period and short-period tracking errors? Moreover, is there some region in which the noise level is low enough to tolerate the high gain of his circuit and can the prediction be made for a length of time which is practicable?

After an extensive series of visits to the commercial laboratories, government and Service organizations, and individuals to obtain information concerning antiaircraft tracking errors, frequency characteristics of input signals, flight errors, etc., sufficient information was obtained to answer the questions raised in the previous paragraph, and a report on this subject was submitted.[1] In it is compared the effectiveness of antiaircraft fire using a fixed memory point predictor, a predictor using a memory point trailing 10 seconds behind the present, and a statistical predictor for certain actual target aircraft courses at Camp Davis. For the actual courses studied, the second method is notably better than the first, but the third is no better than the second.

Thus it seems at present doubtful indeed whether this beautiful mathematical theory has direct practical application to the problem of predicting the future position of aircraft targets. It has produced one valuable result by providing a reference or standard of comparison in terms of which the present prediction methods can be rated. Thus the fact that the very fundamental statistical method yields no significant gain in accuracy over present methods is an important guide for future developments.

It seems highly probable, however, that the theory will have applications to various phases of the general fire-control problem of filtering out signal from noise, and of analyzing the smoothing-prediction problem under various circumstances. It also seems inevitable that this general and powerful analysis will have important applications to other statistical problems.

The project was terminated in February 1943 on the basis of the above stated evidence of the improvement which the statistical method can promise, and on the basis of Service advice concerning the general importance of curved-flight predictors.

5.2.2 Report on the Extrapolation, Interpolation, and Smoothing of Stationary Time Series with Engineering Applications

As a by-product of the work in Project 6, certain general mathematical methods were developed which were believed to be useful in other fields, particularly in statistical work and in the design of electrical filter circuits. Copies of a report[2] covering these phases were distributed as Report to the Services No. 19. This report was the subject of considerable discussion.[3,4,5]

5.3 MECHANICAL AIDS TO COMPUTATION

In several instances during World War II, routine numerical computation became a bottleneck in the carrying out of fire-control testing programs. This focused attention on the possibility of carrying out such computations by means of automatic calculating devices.

The first project to which automatic computation was applied was the preparation of data for an antiaircraft testing device called the tape dynamic tester which has been reported upon by Section 7.1. For the operation of this instrument, it was necessary to have a long paper tape in which is punched a series of holes representing numerical data in an arbitary code scheme, these data being values of one of the coordinates of an aerial target.

In typical cases one computes, by hand, the three coordinates (azimuth, elevation, range) of a fictitious target following a certain desired test course, computing these three data for points, say, 1 second apart on the course. The tape dynamic tester, however, requires these data at intervals of only $\frac{1}{20}$ of a second. Thus a computing device was required which would accept the 1-second interval data, and compute from these, using interpolation formulas involving third differences, 19 intermediate values between every two of the original values.

Since the test courses involve six functions and average about 180 seconds in length, each course involves roughly 20,000 interpolations.

In the life of the project over 60 courses in all were produced, so that well over 1,000,000 interpolations were called for. This represents the equivalent of about 3 years of a skilled computer's time. In addition, the million interpolated numbers were automatically translated into an arbitrary code and holes punched in tape, no errors or corrections being allowed. It is difficult to estimate the time required to do this latter step manually, since even skilled operators found difficulty in going through a long tape without errors.

5.3.1 Relay Interpolator (Project 70)

The foregoing considerations made it desirable to construct a machine capable of interpolating, translating into code, and punching the tapes with a minimum of supervision. Consequently, a device was designed by Division 7 in which the required tasks would be done by telephone relays in accordance with formulas represented by codes punched in a control tape. The schematic and some of the detailed circuits were turned over to the Bell Telephone Laboratories which agreed to build the relay interpolator embodying the schematic; and contract OEMsr-1160 was set up for this purpose with the Western Electric Co., dated July 1, 1943. In 10 weeks the machine was in operation, and continued to be used up to the termination of World War II. In addition to preparing dynamic tester tapes, the relay interpolator has been found useful in many other interpolation problems, and in fact in a rather surprising range of computing problems, whether or not interpolation is involved.

About September 1, 1945, the relay interpolator was moved to the Naval Research Laboratory at Anacostia, where it is now in operation. Several reports have been written on the theory and operation of the interpolator and have been distributed by Division 7 and the Applied Mathematics Panel.[6-9]

5.3.2 Ballistic Computer (Project 74)

A similar situation with regard to computation arose at the Anti-Aircraft Board. The amount of computation to be done in working up test data far exceeded the manpower available. In view of the successful application of relays to the interpolation problem just described, the division decided to use similar means to assist the board.

Again, schematics were prepared by the division. Several new features were required to solve the problems presented. One of these was means for multiplying numbers. Another was means for storing ballistic tables. In place of the circuit for multiplying by repeated addition and shifting as suggested by the division, Bell Telephone Laboratories, under contract OEMsr-1236, chose to use a scheme devised by themselves which employs a multiplication table stored on relays. The storage of tabular values was carried out, as suggested by Division 7, by punching the data and interpolation coefficients on tape, together with the arguments. A hunting circuit causes the tape transmitter to move the tape forward or backward until the desired argument is reached, and the data and interpolation coefficients are read off.

The contract was set up November 12, 1943, and on the night of May 12, 1944, the Anti-Aircraft Artillery Board [AAAB] computer, constructed in accordance with these plans, completed its first unattended run of ballistic computations.

The machine is capable of almost any sequence of computations involving multiplication, division, addition, and subtraction of 5-digit numbers. It will store 10 such numbers simultaneously, and will hold and search through "tables" of functions, each capable of storing about 10,000 digits. The problem data are introduced on two punched tapes, one of which may also be searched. There is practically no limit to the number of steps (addition, searches, etc.) which can be combined into a problem. Ordinary ballistic problems involve 200 to 300 such steps.

In a typical problem, the operator receives the results of a test on an antiaircraft director in the form of a table which shows, usually at 1-second intervals, the three coordinates, range, azimuth, and elevation angle, for the target, and the three gun orders, angle of train, quadrant elevation, and fuze setting, as calculated by the predictor. It is required to compute the error in this prediction.

With the AAAB computer,[10] a girl transcribes the target coordinates and the corresponding times on punched tape, and the gun orders on another tape. Ordinarily, several courses of 200 or 300 sets of data will be transcribed on each tape. The two tapes are then placed in the computer, in which there are tapes carrying ballistic tables and a tape on which are formulas to be used in the computation. The operator pushes the start key and leaves it to the machine to do the rest.

The computing machine reads the target elevation and range at the first recorded instant. It hunts the entry on the ballistic table tape which is nearest this pair of arguments, and interpolates by a quadratic formula in the two variables for the time of flight of a shell to the target. Next, it reads the time at which the data was recorded, and subtracts the time of flight to get the correct firing time.

The computer now needs to know what data the director calculated at the firing time. It refers to the gun order tape and selects the four closest instants, from which it interpolates by a cubic formula for the gun orders transmitted by the director at the firing time. As it obtains the interpolated value for each gun order—fuze, quadrant elevation, and angle of train—it interpolates in the ballistic table to get the correct value, and prints the difference between correct and actual gun orders in the proper column of a result sheet.

The time required by an operator to carry out these computations with the help of the usual commercial calculators is about 40 minutes per set of data. The AAAB computer does the same job in 150 seconds, and works practically 24 hours per day. Thus it does the work of about 50 human operators, with the help of two girls to transcribe data and one part-time maintenance man. If the cost of the machine, about $120,000, is amortized in 3 years (a very conservative figure, since little replacement will be required in 5 or 10 years), then the cost of the man-year of work is well under $1,000 or about ⅓ the cost of manual computation. A more important consideration, however, is that the overall delay in obtaining the results of tests on equipment was greatly reduced.

As a direct outgrowth of the NDRC relay computer development and because of their high order of reliability (experience at Fort Bliss having shown that about one trouble per week would occur on the average, and would require ½ to 1 hour to find and clear), three other computers have been made. The first of these is practically a duplicate of the AAAB computer which is running at the Naval Research Laboratory. Two much larger machines have been designed in consultation with Section 7.1 and are being built for the National Advisory Committee on Aeronautics and for the Aberdeen Proving Ground.

5.3.3 Investigation of the Differential Analyzer (Project 62)

During World War II, the Moore School of Electrical Engineering at the University of Pennsylvania operated its differential analyzer on a full-time basis in the solution of the differential equations of exterior ballistics under direct contract with the Aberdeen Proving Ground. Because of the large amount of work to be done both at Aberdeen and at the University of Pennsylvania, it proved desirable to improve the efficiency of the machines in use. It was hoped that, as a result, manufacturing designs and specifications would be prepared for replacement units which could then be procured directly by the Ordnance Department and the University of Pennsylvania.

A project was thus initiated at the University of Pennsylvania under contract OEMsr-856 and the following program was planned:

1. A study was to be made of the slip occurring in integrators, its sources and means of reducing it in order to operate the units at higher speed. Experimental equipment was built for measuring integrator slip.

2. A study was to be undertaken leading to the immediate application of available and potentially useful torque amplifiers. The ones used in the analyzer, although satisfactory under ordinary conditions, were not able to

perform under conditions of full-time employment. The maintenance required was excessive and the accuracy of the integrators impaired.

3. The preceding two items were hopefully intended to raise the operating level of the machine by a factor of two or three. If and when this was done, it would be necessary to take into account the effect of increased speed on other units. A new counter tripping unit would be required to provide compensation for the time lag in the counter system. Also, several special input tables, then in use and operating at the lower speed level, would have to be refined in order to operate successfully at higher speeds.

4. The development of special data-transfer equipment to permit use of improved ballistic methods was to be undertaken. This item was essential to the efficient utilization of otherwise unused integrators.

During the work doubt arose as to the accuracy of the experimental setup for measuring integrator slip. It was therefore decided to review this aspect of the program very carefully to determine what useful results, if any, could be obtained with the existing setup; what problems relating to integrator slip appeared to be most important; and what experimental setup would apparently be necessary to solve those problems.

Under item 2, the then available types of torque amplifiers, including a Polaroid type developed by the General Electric Co., a modified Polaroid type, a Maxson hydraulic unit, and the torque amplifier used in the M5 antiaircraft predictor, were tested. None of them was found satisfactory for the particular application. In some instances the torque output was insufficient; in some instances the maximum speed was too low; in some instances the input torque was too great; and in some instances the torque amplifiers were unstable when connected in series.

The contract was thus extended in time, and modestly supplemented in funds, for the purpose of finishing up those aspects of the integrator slip problem which could be handled with the experimental setup, and to make possible the completion of the survey of torque amplifiers for use in differential analyzers. Changes were made (chiefly involving an improved optical system) in the Polaroid type of torque amplifier referred to above. At the same time progress was made in improving the bands and strings for the Niemann type of torque amplifier now used on the differential analyzers at the Massachusetts Institute of Technology (that is, on the older model), at Aberdeen, at the University of Pennsylvania, and elsewhere.

The modified torque amplifier aspect of the study was brought to a successful conclusion, and a contract with the Army negotiated under which eight servos of the new type were constructed to carry out practical tests on the differential analyzer. With exhaustion of contract funds, certain interim work was carried on by the University of Pennsylvania, pending the arrangements for the new contract with the Ordnance Department. This interim work was devoted to developing a curve-following mechanism capable of speeds comparable with that of the amplidyne-Polaroid servomechanisms previously developed. On tests, the photocell-controlled mechanism operated satisfactorily when following the edge of a line ⅛ inch wide.

Further details are available in the final report under the contract.[11]

5.4 MISCELLANEOUS PROBLEMS

The services of two mathematicians[b] were made available by placing contracts with the Princeton University (contract NDCrc-105) and the University of Wisconsin (contract NDCrc-116). Work on the latter was interrupted, and the contract was terminated before definitive results were obtained. Work on the former resulted in five studies:

Some Experimental Results on the Deflection Mechanisms.[12] This is a detailed study of the accuracy and stability of a deflection mechanism produced by the fire-control design group at the Frankford Arsenal. In connection with this study extensive use was made of the differential analyzer at the Massachusetts Institute of Technology.

[b] C. E. Shannon and I. S. Sokolnikoff.

Backlash in Overdamped Systems.[13] The preceding report[12] showed that backlash could cause sustained oscillation in a second-order mechanical system, provided that it was less than critically damped. This paper attacks the same problem for overdamped systems.

The Theory of Linear Differential and Smoothing Operators.[14] This is a general study of predicting and smoothing mechanisms.

A Height Data Smoothing Mechanism.[15] This is an analytical study of a particular mechanism which might be used for the smoothing of height data.

The Theory and Design of Linear Differential Equation Machines.[16] This paper gives a general mathematical theory which permits the rapid analysis of mechanical computing circuits and the design of mechanical computing circuits having desired characteristics.

5.5 PROJECTS TRANSFERRED TO THE APPLIED MATHEMATICS PANEL

5.5.1 Statistics of Train Bombing (Project 23)

This project was begun in July 1941 under contract OEMsr-817, originally administered by Princeton University. Subsequently the work was expanded to include a contract with the University of California, and the Princeton contract was expanded and shifted to Columbia University. The project originated in a request for the design of a bombardier's calculator, but subsequently developed into a broad study of the statistics of train bombing.

Basic tables for the probability of at least one hit on various rectangular targets were calculated, showing the way in which this probability depends upon the number of bombs in the train, the bomb spacing, the angle of approach, the size and proportion of the target, and the magnitude of the aiming and dispersion errors.

Extensive studies were carried out to extend these results to produce a general theory of multiple hits on multiple targets. This theory analyzes the way in which missions should be planned and attacks carried out in order to maximize the number of cases in which at least k hits ($k=1$ to 5) will be obtained.

Studies were made to determine whether or not these theories could be usefully applied under circumstances where (as must always be the case) the aiming errors are not actually known. The theory of multiple hits was checked by a study of a train-bombing experiment at Eglin Field, and also by an experiment carried out for this purpose on a bombsight trainer.

Studies were under way at the date of transfer of the project to the Applied Mathematics Panel to determine the optimum type of attack on maneuvering targets. Five reports were issued under this project.[17-21]

5.5.2 Computations (Project 39)

In connection with a variety of projects and plans, the division found it increasingly necessary to have computations performed. These were both of routine and of highly specialized character. In each case it was most efficient to have the actual work performed at the particular place best equipped to handle it. This project was begun under the auspices of the Franklin Institute under contract OEMsr-444, enabling the division to carry out computations whenever the necessity arose by requesting the Franklin Institute to procure the work from the most effective agency.

The contract proved exceedingly useful. Under it were supported a study of the differential analyzer-ballistics problem, a study of the fragmentation-damage problem, a study of scatter bombing, etc. Since formal reports were not required under the terms of the contract none was submitted. The reader should therefore turn to the Summary Technical Report of the Applied Mathematics Panel for a bibliography.

5.5.3 Air Warfare Analysis (Project 47)

As a result of a conference held by the Fire-Control Division with representatives of the Naval Bureau of Ordnance, the Naval Bureau of Aeronautics, the Office of the Coordinator of Research of the Navy, and the Army Air

Corps, there was planned a general program for a series of probability and statistical studies of plane-to-plane fire. It was hoped that such a study, by evaluating the influence of the various factors on the overall effectiveness of plane-to-plane fire, would give some practical information on certain problems of design of airborne fire-control systems. Accordingly contract OEMsr-618 was negotiated with Columbia University for this purpose.

At a later conference with Service personnel it was agreed that the group could most usefully first attack the problem of estimating, through probability considerations, the comparative effectiveness of various mixed batteries for a fixed gun fighter attacking a bomber. Those connected with the project visited the AAU at Norfolk and had firsthand contacts with aircraft fire-control equipment.

Early in 1943 the project was expanded in personnel and broadened in scope so that it could undertake any studies under the general title of Air Warfare Analysis. The group was then composed of about fifteen technically trained people. Among the studies that were undertaken as a result of direct requests from the Services or other NDRC divisions were: statistical acceptance tests for bombsights; analysis of a dive bombsight; estimate of additional risks to a bomber due to extensions of the straight bombing run; probability of damage to a dive bomber; optimum ammunition for air combat, and counter-evasion measures for aerial torpedoing; problems bearing upon the statistical aspects of the testing of certain naval antiaircraft fire-control equipment; and problems relating to the broad aspects of probability of damage to aircraft through antiaircraft fire, plane vulnerability, optimum interrelations of aiming errors and gun dispersions, etc.

For a detailed account of this work the reader is referred to the Summary Technical Report of the Applied Mathematics Panel.

Chapter 6

SEABORNE FIRE CONTROL WITH RADAR

Section 7.6, charged with Navy fire control with radar, was formally organized very late in World War II (January 1944). Thus in view of the late start, the researches and developments undertaken after formal section organization necessarily were destined for fruition after the cessation of hostilities. In view of its field of activity the responsibilities for projects involving radar already underway were inherited by the section at the time of its formation. Inasmuch as the Chief of Section 7.6 was a member of Division 7 from the date of NDRC reorganization, the four or five projects begun before January 1944, which were his responsibility, might be said to carry this date, informally, back another year. Thus, despite the technical accomplishments which are to be recorded below it was the feeling of the section that its contribution towards winning the war was nil.

In turning to the technical scene, one must return to the days of Section D-2 to summarize the radar developments sponsored by the division.

6.1 DEVELOPMENT OF THE RADAR SCR-547 (PROJECT 14)

During the early months of World War II, optical range and height finder equipment was a necessity and had no real competition in terms of equipment "in being" which met military requirements for supplying fire-control data. It was considered essential during the interim period of radar development to bring existing optical instruments to the highest possible level of performance. Furthermore, elementary consideration of prudence demanded that the situation be strongly hedged against two possibilities: (1) that unforeseen development, production, or training difficulties might seriously delay the application of radar as a means for obtaining fire-control data under the highly mobile conditions of field use; (2) that the enemy might develop effective countermeasures. Hence Section D-2 vigorously pursued a program of optical range finder development. (See Section 2.6, and also Volume 2.)

But Section D-2 did not rest with a program of mere "watchful waiting" with respect to radar. In January 1941, a request was made to Section D-1 for the development of radar equipment to provide range data for antiaircraft fire-control systems. In view of saturation of the facilities of the Radiation Laboratory at the Massachusetts Institute of Technology, Section D-1 recommended that Section D-2 undertake to develop a range-only radar under separate contract with another laboratory.

Accordingly, there was developed at the Bell Telephone Laboratories under contract NDCrc-156, a 10-cm transmitter pulsing at 400 cycles per second, and transmitting from a 54-in. parabolic antenna. A second 54-in. parabola receives the reflections. The antennas and other radio equipment are mounted on a modified M2A4 sound-locator trailer with telescopes and controls for tracking in azimuth and elevation. Seats are provided for trackers and range operator (Figure 1). Because of the physical appearance of this device it was nicknamed "Mickey Mouse."

Range was measured by means of calibrated phase shifters which position the reflection with respect to a "step" in the timing trace on a cathode-ray screen. A full-range scale using a 4,000-c sweep displays all reflections from the 2,500-yard minimum to 41,600 yards. For precision measurements a 100,000-c sweep is used. This sweep voltage is obtained from a phase shifter which is geared mechanically (25:1 ratio) to the phase shifter operating at 4,000 c so that the revolutions of the 100-kc phase shifter are counted with respect to the phase of the pulsing frequency. The readings of dials mounted on the two phase shifters are combined to read range, or the positions may be transmitted to coarse-fine selsyns.

The experimental equipment was tested at Fort Monroe during the period from July 13, 1941 to August 20, 1941. These tests included range measurements on fixed targets, surface vessels, casual aircraft, and target planes flying missions. Details of these tests are summarized in the report of the Coast Artillery Board on Project 1213, August 23, 1941. In general, the results indicated that the probable error of range measurement with the optically tracked radio equipment was about 1/3 that of a stereoscopic height finder. There was, of course, the further advantage that the range data from the range finder are reasonably smooth so that a good range rate is available.

This instrument was subsequently standardized by the Army as the radar SCR-547.[1]

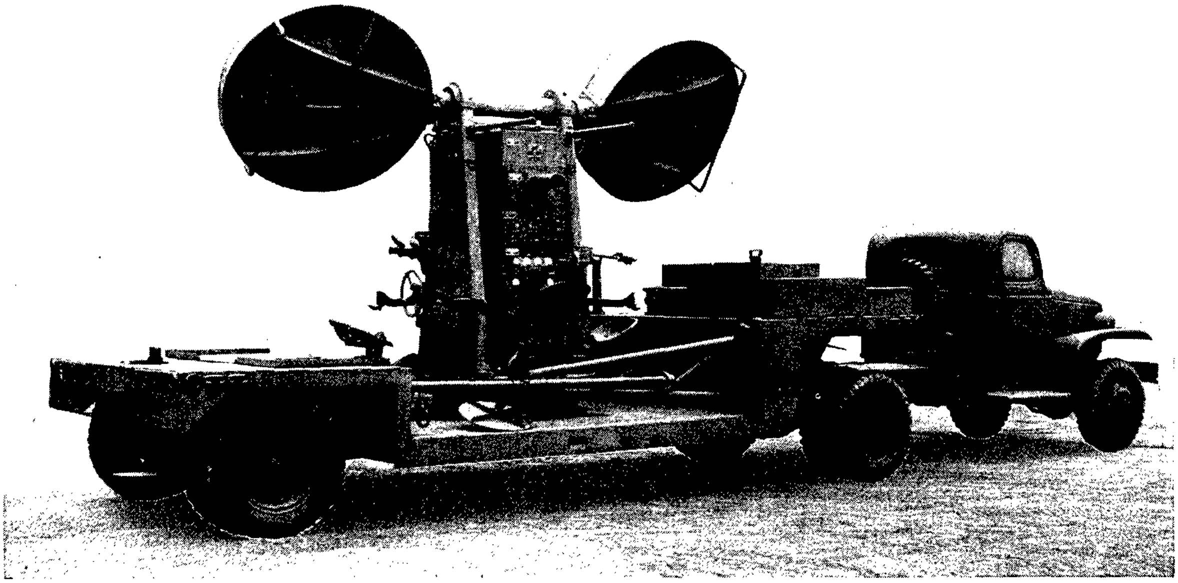


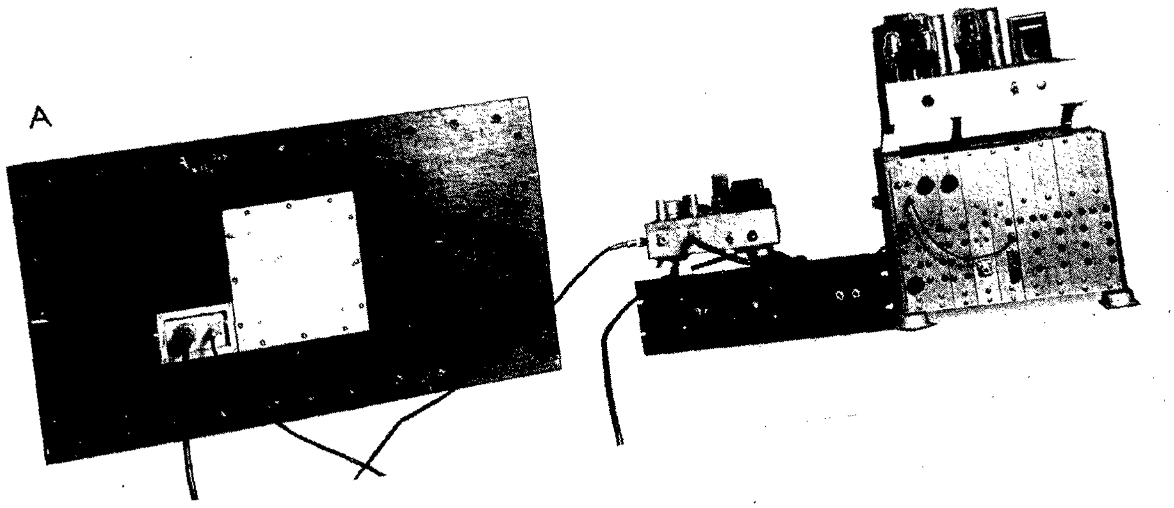
FIGURE 1. SCR-547 antiaircraft range-finding equipment.
This radar is located on a trailer and requires three operators. Men seated on the left and right of the apparatus use telescopes to keep the two parabolic antennas on the target. A third man is the radar operator. For obvious reasons this device was nicknamed "Mickey Mouse."

6.2 DEVELOPMENT OF THE CHRONOGRAPH T4 (PROJECTS 65 AND 83)

In December 1942 a suggestion was advanced that Division 7 sponsor a study to determine the practicability of measuring the velocity of 90-mm projectiles near the muzzle of the gun by means of the Doppler effect with continuous-wave radar. In January 1943 contract OEMsr-983 was negotiated with the Westinghouse Electric and Manufacturing Co. for this purpose.

This study showed that portable field chronographs could be made based on this principle.[2] Under contract OEMsr-1405 (Project 83) 17 engineered units were built for the Armed Services.[3]

The general scheme was as follows: Microwave energy is projected along the trajectory near the muzzle from a paraboloid antenna located near the gun. Some of this energy is intercepted and re-radiated by the projectile in flight. Because of the Doppler effect the frequency of the reflected energy differs from that of the energy radiated by the antenna by an amount proportional to the projectile velocity and the frequency of the radiated energy. Part of the reflected energy is picked up by a receiving paraboloid located adjacent to the transmitter. This energy is conveyed to a crystal mixer where it is mixed with some of the directly transmitted energy to produce a voltage of Doppler frequency which is a measure of projectile velocity. This voltage is amplified



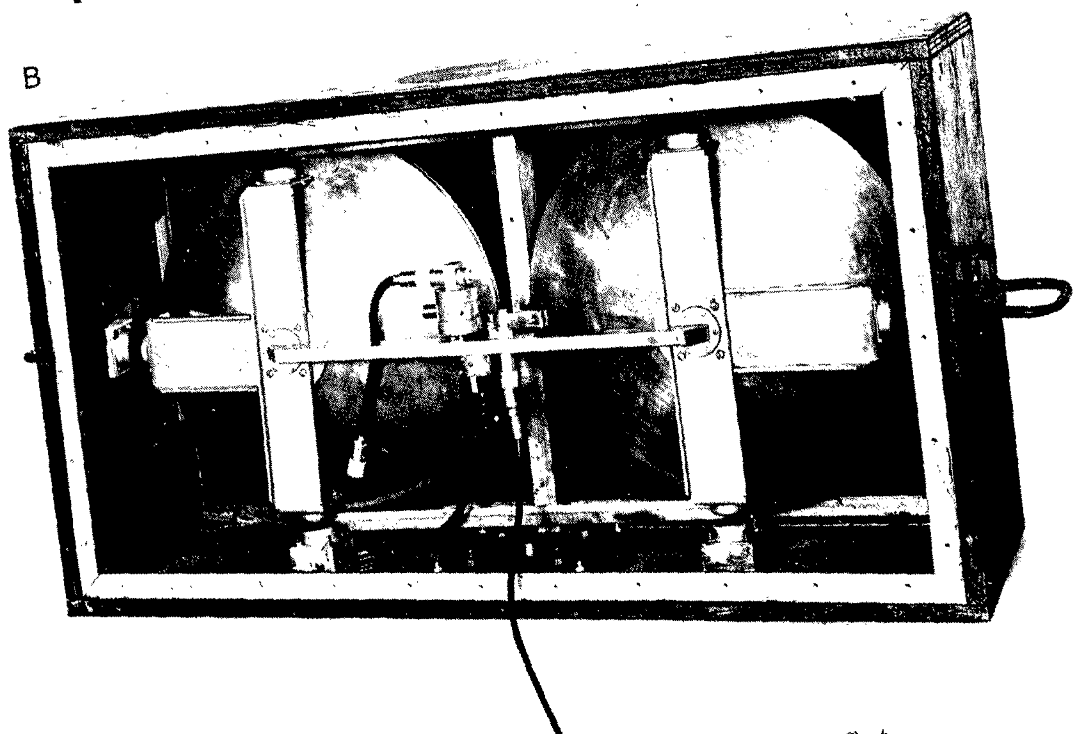


FIGURE 2. T-4 chronograph utilizing the Doppler effect.

This radar chronograph transmits an r-f signal which is reflected by a moving projectile. The received frequency is heterodyned with the transmitter frequency in the unit to produce the Doppler frequency. The time duration of an arbitrarily selected number, or train, of Doppler cycles is measured by special frequency and time-counting circuits. The data thus obtained is converted into velocity in feet per second by a simple calculation. A shows the r-f unit containing two paraboloid antennas to the left, the power supply unit in the center, and the counter unit to the right. B shows the r-f unit with cover removed showing the two paraboloid antennas.

and then recorded by an electronic counter (Figures 2A and 2B).

For the purpose of the investigation, radiation of 3-cm wavelength is used. For a projectile velocity of 2,850 ft per sec this gives a Doppler frequency of approximately 58,000 c. This frequency is amplified. A finite train of say 256 cycles actuates a counter. During this interval another electronic counter counts the number of cycles of a 200-kc crystal oscillator. The number of 200-kc counts during the one Doppler-frequency train is inversely proportional to the Doppler frequency. A small correction for geometry is required to give the actual muzzle velocity.

Preliminary tests indicated that two 15-in. parabolas, one transmitting 20 milliwatts of power, are adequate for ranges up to 600 feet on 90-mm shells. Oscillograms were obtained which showed that the limits of accuracy of the measurement of velocity are set only by the accuracy of measurement of the transmitter frequency and the Doppler frequency. With the counter method of recording, ¼ per cent accuracy was obtained.

The effects of gun flash and shock were investigated and satisfactory tests made at Aberdeen and Dahlgren.

Under contract OEMsr-1404 with the Baltimore Plant of the Westinghouse Electric and Manufacturing Co., 17 experimental models of the chronograph T4 were constructed.[3] These were delivered to the U. S. Army and Navy, to the British, and to the Civil Aeronautics Administration.

One of the production models, prior to its acceptance, together with the original laboratory model was taken by the Bureau of Ordnance to the Pacific for measurements on the muzzle velocity of 16-in. guns aboard battleships. Successful and accurate measurements of the muzzle velocity were realized on about two-thirds of the rounds fired. Weaknesses in the production prototype were discovered and corrected in all the production units. The only serious difficulty encountered in these tests aboard ships was the triggering of the counting circuits by X-band radars. It was found necessary to turn off the ship's X-band radars as well as the X-band radars of all other ships in the vicinity during the tests. The contractor recommended new development at a much shorter wavelength, where no radars are now contemplated. This recommendation was passed on to the Services.

6.3 SEABORNE TORPEDO DIRECTORS (PROJECT 72)

Following the development of the Mark 32 airborne director (see Volume 3), the Navy requested Section 7.2 to adapt that development to the motor torpedo boat. The official request (NO-134) specified a director with a course and speed sight and also with blind-firing equipment. The coordination of this project with the Radiation Laboratory under Division 14 disclosed the development of a simplified device, perhaps best described as a mechanized maneuvering board. It was agreed at a meeting of representatives of Divisions 7 and 14 that it would be well to have two independent attacks on the problem. The Navy concurred and contract OEMsr-1208 was negotiated with the General Electric Co.

Under the project a preliminary working model with a mechanical computer of the simplest possible type was constructed.[4,5] This computer permitted firing of the torpedo from a motor torpedo boat at any time the captain of the boat wished to bring the direction of his own course along a continuously calculated value. This director permitted avoiding action and firing on the run. Tests at Miami, Florida, indicated that the predictions were quite erratic. Several sources of error existed, such as the radar, the data-transmission system from the antenna to the PPI, the compass, the autosyn transmission of own-ship's course, and the flexible shafts from the computer to the radar. No improvement to the system could be made until most of these errors were eliminated.

A program was instituted to remedy the sources of error: first, by the use of K-band radar; and secondly, by redesign of those portions of the system which had shown up as sources of error in the Florida tests. Although work had begun on the assembly of the new system, the conclusion of World War II prevented completion of the project. In conformity with the termination policy as set up by OSRD, the Bureau of Ordnance [BuOrd] was given an opportunity to take over this project. BuOrd indicated that it wished to review its entire program in the light of changed circumstances brought about by the end of war and that they deemed it desirable to reconsider the entire fire-control program for motor torpedo boats. The Navy will undertake, therefore, to set up a project at the Naval Research Laboratory for further mathematical analysis of the problem. It was agreeable to the BuOrd that this project be terminated. The project was therefore terminated uncompleted on September 30, 1945.

BuOrd also asked NDRC in NO-197[4] to develop a better torpedo director for destroyers than the Mark 27 which was then in use and to make it blind firing. Examination of the problem led to the conclusion that the director designed for motor torpedo boats was not adequate for a destroyer, and that more experiments would be needed. Theoretical studies were made to devise a director which would not be too cumbersome and large but would have the required accuracies. A satisfactory solution was found and design work was begun. With the termination of the contract, the work was continued under a direct BuOrd contract with the General Electric Co.

6.4 REDESIGN OF GUN DIRECTOR MARK 49

When director Mark 49 was placed in service unfortunate failures of some component parts resulted in the cancellation of the contracts for its production. Upon Navy invitation the section made a study which indicated that if certain changes were made the director would probably be an acceptable piece of naval equipment.

One of the outstanding difficulties with the Mark 49 was the failure of the clutch-type power drive after a relatively small number of hours of operation. The take-off of the rate-measuring gyro was crude and entirely unsatisfactory in operation. There were also a number of other bad features such as restricted angle of vision of the operator and maximum lead angle, inadequate radar reflector, and poor radar indication.

Under the original request it was understood that NDRC would attempt to make whatever modifications were necessary in the gun director Mark 49 system to increase its usefulness as a blind-firing unit. As a result of subsequent discussions the program was narrowed in scope, so as to include only modifications in the power drive and gyro element.

Under contract OEMsr-1235 the Servomechanisms Laboratory at the Massachusetts Institute of Technology installed amplidyne power drives in one unit. The contactor take-off on the gyro was replaced by an induction take-off.[6] The resulting system has a maximum angular rate in train of 35 degrees per sec and maximum acceleration in the neighborhood of 100 degrees per sec per sec. A modified gun director Mark 49 was delivered to the Naval Test Center at Dam Neck, Virginia.

In spite of the very satisfactory tracking and drive characteristics obtained in the modified director, these modifications were not put into Service use largely because the number of Mark 49 directors for which such modification procedure was practicable was too small to justify the cost.

6.5 GUNFIRE-CONTROL SYSTEM MARK 56 (PROJECTS 71, 79, 85)

At the beginning of 1944 the record of the Navy fire-control development in antiaircraft was as follows: (1) in the field of heavy antiaircraft guns no new fire-control equipment had been introduced into the Navy since the beginning of World War II; (2) in the field of automatic weapons, the director Mark 51 had been introduced and some improvement was on the way; (3) the heavy antiaircraft gun directors employed no radar with a beam less than about 16 degrees in width; and (4) no microwave equipment was getting into the fleet for one reason or another except as an antiaircraft set for use with directors Mark 33 on certain cruisers. It was abundantly clear to Division 7 members and their employees that the Navy was in need of a modern radar-equipped director capable of employing radar data, of getting quick solutions, and of performing the calculations with an accuracy adequate over all ranges of speeds of modern planes. It was under these circumstances that the NDRC project known as Gunfire-Control System Mark 56 was started. (See Figures 3 and 4.)

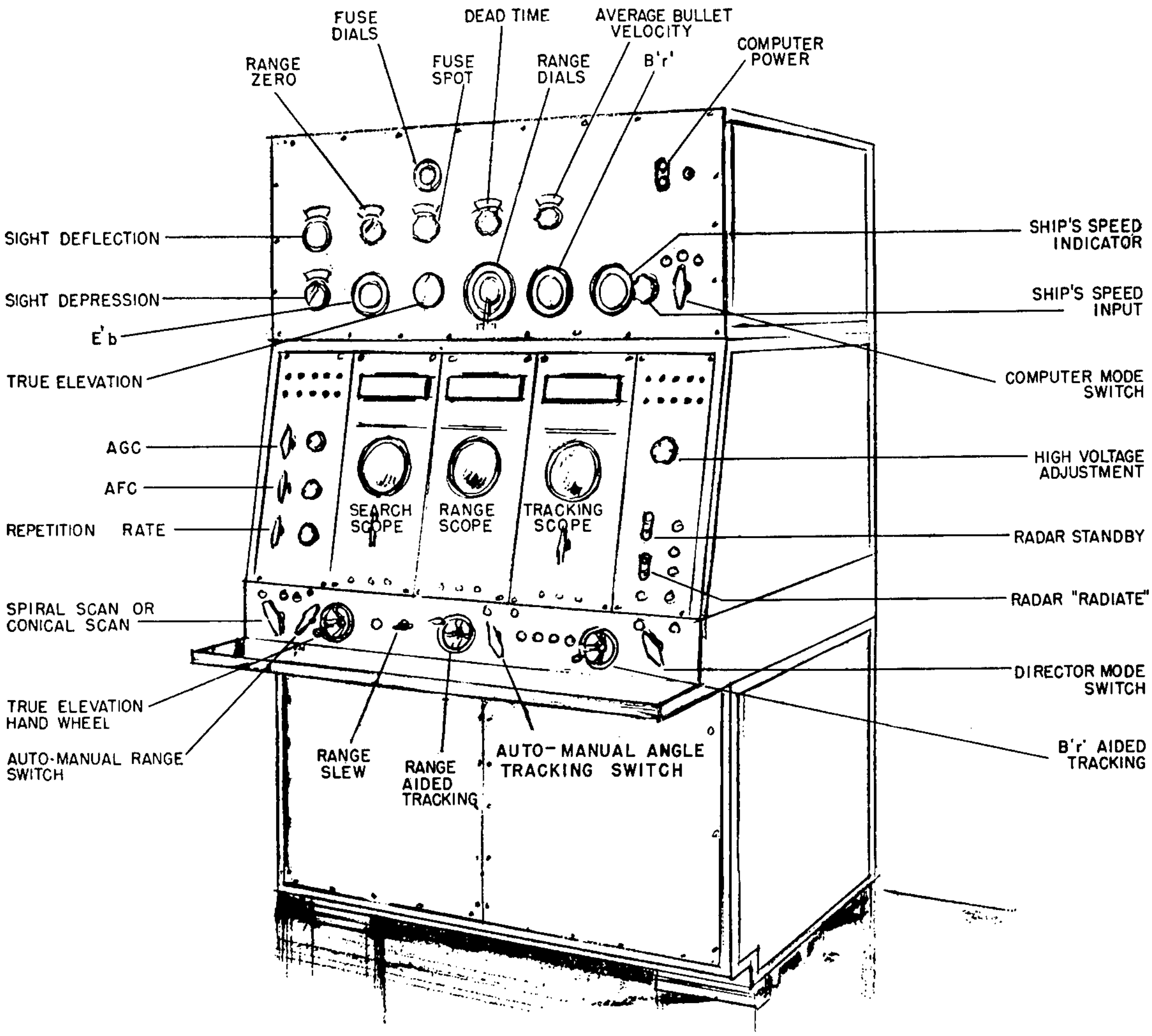


FIGURE 3. Console of Mark 56 Gunfire-Control System. This unit of the Mark 56 is below deck and represents the nerve center of the system.

It is desirable to state briefly the history of the gunfire-control system project up to the time of the formation of Section 7.6 of NDRC.[7] The failure of NDRC to provide an adequate antiaircraft fire-control system for the Navy including radar, by patching up a system already in existence in the Navy, convinced the members of the Radiation Laboratory and members of Division 7 of NDRC that a fully integrated design of a complete fire-control system was required. Great fear was expressed in many quarters that the enemy might develop aerial weapons with which neither the director system Mark 37 nor its radars could cope. Section T of OSRD that had just successfully completed the proximity fuze development, apparently reached the same conclusions at about the same time. Therefore, to assist the Navy, a deliberate program was established by

NDRC to provide a completely integrated radar fire-control system. This project was to be carried through to complete engineering design, the building of manufacturing prototypes, the furnishing of complete manufacturing drawings, and the setting up of a manufacturing source and all necessary vendors. This approach was necessitated by the past experiences in which laboratory prototypes had been found wanting not operationally, but because of the long time element between laboratory research and their subsequent engineering for production. The plan required the following steps:

1. Complete support from the heads of Divisions 7 and 14 of NDRC.
2. Complete cooperation of the Director of Radiation Laboratory.
3. The sympathetic understanding by the heads of NDRC and OSRD.
4. The establishment of adequate contracting with a company able to assist in the engineering and also able to manufacture.
5. The sponsorship and cooperation of the Bureau of Ordnance.

The Bureau of Ordnance indicated its support by requesting the establishment of the project NO-166 in a letter, dated May 18, 1943.

The General Electric Co. was brought into the picture by an NDRC contract, first for the development of a suitable gyro unit (OEMsr-1181), and later for supplying engineering, parts, and the construction of two complete director systems (OEMsr-1299).

Under contract OEMsr-1181 the General Electric Co. cooperated in the development of a line-of-sight gyro and a vertical gyro which supplied both stabilization and regular rate data for the gun director Mark 56.[8]

The gun director Mark 56 (Figure 3) is for use primarily with 5-inch dual-purpose guns with emphasis on low-flying torpedo plane attacks. It is a fully automatic blind-firing unit with emphasis on ruggedness, ease of maintenance, and speed of response.

As noted above, this development was sponsored cooperatively by Divisions 7 and 14 of NDRC. Development work was done both at the Radiation Laboratory under contract OEMsr-262 and by the General Electric Co. under contract OEMsr-1299. Component work was done at both laboratories and a first laboratory unit designated gun director Mark 56 Model O was set up at the Radiation Laboratory. Two additional laboratory models were built, followed by two prototypes from manufacturing drawings. Under the terms of the contract the General Electric Co. built these two complete prototype directors based on the laboratory models.

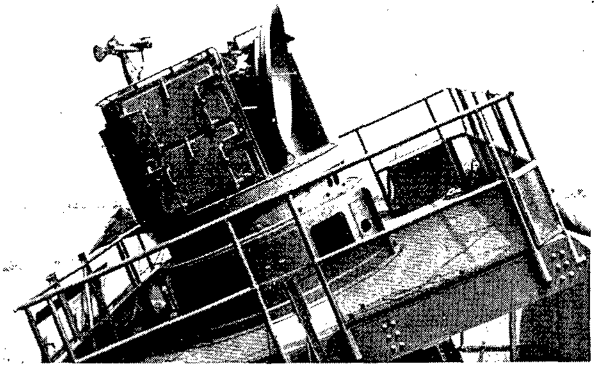


FIGURE 4. Antenna of the Mark 56 radar. The X-band radar is fully automatic in all coordinates. The stabilization of the disk is apparent in the figure by the horizontal position of the antenna feed pointing to the horizon in contrast to the tilt of the deck.

The gun director Mark 56 employs a two-axis mount.[9] Rate stabilization and tracking is performed through a gyro unit assembly. This unit fixes rotation around a line of sight giving, so far as prediction is concerned, a three-axis system. Bullet velocity is computed explicitly and is used in calculating angular deflections. The angular deflections are added to the present position and the sums are converted to future gun coordinates by a mechanical system of bevel gears.

An X-band radar fully automatic in all coordinates (Figure 4) is used. Spiral scan is incorporated. Emphasis was placed on antijamming features and on built-in test equipment.

The termination policy of OSRD did not permit the completion of this program. It was therefore transferred to BuOrd, which negotiated a new contract with the General Electric Co. for the completion of the units 4 and 5, for which it had prime responsibility. The contract of BuOrd became effective on October 28, 1945, and the NDRC project terminated on that date. The new contract is of the task type, under which provisions are made for coordinating the work at the Librascope Corp. (ballistic computer), and for providing assistance to the Navy in maintaining the two units for which the Radiation Laboratory had prime responsibility (OEMsr-262). The two units which the Radiation Laboratory delivered to the Navy were physically complete, but not fully "debugged" or tested.

Prior to the end of World War II, the Navy had placed a letter of intent with General Electric for the construction of a substantial quantity of the Gunfire-Control System Mark 56. The cessation of hostilities did not bring about a cancellation.

Contract OEMsr-1044 with the Librascope Corp. was transferred from Division 14 to Division 7 when it became evident that the bulk of this contract covered a part of the computer for the gun director Mark 56. The portion of the computer which Librascope Corp. manufactured was called the ballistic computer.[10] It is a mechanical computer using levers for addition, multiplication, division, and ballistic functions. Its inputs are driven by remote transmission from the gun director Mark 56 and its associated radar Mark 35. It uses these observed quantities to compute certain functions such as u, the average bullet velocity to future position; g, the time of flight were there no cross components of the target velocity; and various other functions. The outputs of the computer are voltages. The voltages are used in electrical networks solving for gun deflections, unit parallax, and wind corrections. Hand inputs to the computer provide for true wind, dead time, initial velocity, density, and spot correction. The computer is not complete by itself. On the other hand, because of the integration of the gun director Mark 56 system, it contains parts of the radar and data-transmission systems.

A breadboard model of the computer was delivered and tested for Class A errors as well as for effects of vibration. Using these tests as checks, detailed specifications were set up with Librascope Corp. to guarantee the success of the computer from all Service standpoints.

Five prototype models were required by this contract.

In order to provide proper design and integration of this unit with the rest of the Mark 56 system, personnel from the General Electric Co. under contract OEMsr-1299 and the Radiation Laboratory under contract OEMsr-262 were made available to the Librascope Corp.

This project was terminated by NDRC as of October 31, 1945. BuOrd placed a new contract, effective the same date, for the continuation of the subject work. In order to provide for engineering coordination with the General Electric Co., which is to be responsible for the overall system, the new contract instructed the Librascope Corp. to deal directly with the General Electric Co. on engineering matters and with BuOrd on legal and fiscal matters.

PART II

DATA SMOOTHING AND PREDICTION IN FIRE-CONTROL SYSTEMS

By *R. B. Blackman, H. W. Bode,* and *C. E. Shannon* [a]

THE PROBLEM of data smoothing in fire control arises because observations of target positions are never completely accurate. If the target is located by radar, for example, we may expect errors in range running from perhaps 10 to 50 yards in typical cases. Angular errors may vary from perhaps one to several mils, corresponding at representative ranges, to yardage errors about equal to those mentioned for range. Similar figures might be cited for the errors involved in optical tracking by various devices. Evidently these errors in observation will generate corresponding errors in the final aiming orders delivered by the fire-control system.

A data-smoothing device is a means for minimizing the consequences of observational errors by, in effect, averaging the results of observations taken over a period of time. The simplest example of data smoothing is furnished by artillery fire at a fixed land target. Here the principal parameter is the range to the target. While individual determinations of the range may be somewhat in error, a reliable estimate can ordinarily be obtained by taking the simple average of a number of such observations. This example, however, is scarcely a representative one for problems in data smoothing generally. The errors involved are small and the averaging process is an elementary one. Moreover, the data-smoothing process is not of very decisive importance in any case, since any errors which may exist in the estimated range can normally be wiped out merely by observing the results of a few trial shots.

More representative problems in data smoothing arise when we deal with a moving target. In this case errors in observational data may be much more serious, since they determine not only the present position of the target but also the rates used in calculating how much the target will move during the time it takes the projectile to reach it. An illustration is furnished by antiaircraft fire against distant airplanes. Suppose, for example, that in observing the target's position we make two errors of opposite sign and a second apart, of 25 yards each. Then the apparent motion of the airplane is in error by 50 yards per second. Since the time of flight of an antiaircraft shell in reaching its target may be as high as 30 seconds or more, such an error might produce a miss of the order of 1 mile. It is clear that in any comparable situation the effect of observational errors in determining the target rate will be much greater than the position error alone would suggest, and the function of the data-smoothing network in averaging the data so that even moderately reliable rates can be obtained as a basis for prediction becomes a critically important one.

Aside from magnifying the consequences of small errors in target position, the motion of the target complicates the data-smoothing problem in two other respects. The first is the fact that it gives us only a brief time in which to obtain suitable firing orders. The total engagement is likely to last for only a brief time, and in any case it is necessary to make use of the data before the target has time to do something different. Thus the averaging process cannot take too long. The second complication results from the fact that the true target position is an unknown function of time rather than a mere constant. Thus many more possibilities are open than would be the case with fixed targets, and the problem of averaging to remove the effects of small errors is correspondingly more elusive.

The intimate relation between data smoothing and target mobility explains why the data-smoothing problem is relatively new in warfare. The problem emerged as a serious one only recently, with the introduction of new and highly mobile military devices. The airplane is, of course, the archetype of such mobile instruments, and we have already mentioned the data-smoothing problem as it appears in antiaircraft fire. Since the relative velocity of airplane and ground is the same whether we station ourselves on one or the other, however, the

[a] Bell Telephone Laboratories.

mobility of the airplane produces essentially the same sort of problem in the design of bombsights also. Another field exists in plane-to-plane gunnery. Although they are somewhat slower, the mobility of such vehicles as tanks and torpedo boats is still considerable enough to create a serious problem here also. Future examples may be centered largely on robot missiles. It is interesting to notice that a guided missile may present a problem in data smoothing either because it belongs to the enemy, and is therefore something to shoot at, or because it belongs to us, and requires smoothing to correct errors in the data which it uses for guidance. The tendency to higher and higher speeds in all these devices must evidently mean that fire control generally, and data smoothing as one aspect of fire control, must become more and more important, unless war making can be ended.

Very mobile instruments of war, such as the airplane, began to make their appearance in World War I, but there was insufficient time during that war to make much progress with the fire-control problems which such instrumentalities imply. In the interval between World War I and World War II, however, a considerable number of fire-control devices, such as bombsights and antiaircraft computers, were developed. The principal attention in the design of these devices, however, was on the kinematical aspects of the situation. Although a number of them included fairly successful methods of minimizing the effects of observational errors,[b] it seems fair to say that in the interval between the two wars there was no general appreciation of the existence of the data-smoothing problem as such.

It follows that the theory of data smoothing advanced in this monograph is the result principally of experience gained in World War II. More specifically, it is the product of the experience of the authors with a series of projects, largely sponsored by Division 7 of NDRC, concerned with the design of electrical antiaircraft directors. In addition, it draws largely on the results of a number of other investigations, also NDRC sponsored. The possible key importance of data smoothing in the design of fire-control systems was recognized by Division 7 early in the course of its activities and the emphasis placed upon it in a number of projects led to the accumulation of a much larger body of results than might otherwise have been obtained.

Data smoothing is developed here in terms of concepts familiar in communication engineering. This is a natural approach since data smoothing is evidently a special case of the transmission, manipulation, and utilization of intelligence. The other principal, and perhaps still more fundamental, approach to data smoothing is to regard it as a problem in statistics. This is the line followed in the classic work[1] by Norbert Wiener.[c] For reasons which are brought out later, Wiener's theory is not used in the present monograph as a basis for the actual design of data-smoothing networks. Because of its fundamental interest, however, a sketch of Wiener's theory is included. The authors' apologies are due for any mutilation to the theory caused by the attempt to simplify it and compress it into a brief space.

The present monograph falls roughly into two dissimilar halves. The first half, consisting of the first three or four chapters, includes a discussion of the general theoretical foundations of the data-smoothing problem, the best established ways of approaching the problem, the assumptions they involve, and the authors' judgment concerning the assumptions which best fit the tactical facts. In this part may also be included the last chapter, which contains a fragmentary discussion of alternative data-smoothing possibilities lying outside the main theoretical framework of the monograph.

The rest of the monograph is concerned with the technique of designing specific data-smoothing structures. A fairly elaborate and detailed treatment is given here, in the belief that the

[b] Most of these solutions depended upon the use of special types of tracking systems. Examples are found in the use of regenerative tracking in bombsights and antiaircraft computers or in the determination of rates from a precessing gyroscope or an aided laying mechanism in an antiaircraft tracking head. So far as their effect on the data-smoothing characteristics of the overall circuit is concerned, these devices are equivalent to simple types of smoothing networks inserted directly in the prediction system. This is discussed in more detail under the heading "Exponential Smoothing," Section 10.1.

[c] Wiener is also responsible for providing tools which permit the gap between the statistical and communication points of view to be bridged.

problem of actually realizing a suitable data-smoothing device is, in some ways at least, as difficult as that of deciding what the general properties of such a device should be. The technique, as given, draws heavily upon the highly developed resources of electric network theory. For this reason the discussion is couched entirely in electrical language, although the authors realize, of course, that equivalent nonelectrical solutions may exist. For the benefit of readers who may not be familiar with network theory, the monograph includes an appendix summarizing the principles most needed in the main text.

Two further remarks may be helpful in understanding the monograph. The first concerns the relation between data smoothing and the overall problem of prediction in a fire-control circuit. These two are coupled together in the title of the monograph, and it is clear that the connection between them must be very close, since, as we saw earlier, small irregularities in input data are likely to be serious only as they affect the extrapolation used to determine the future position of a moving target. In the statistical approach, in fact, data smoothing and prediction are treated as a single problem and a single device performs both operations.

In the attack which is treated at greatest length in the monograph a certain distinction between data smoothing and prediction can be made. To simplify the exposition as much as possible, the explicit discussion in the monograph is directed principally at data smoothing. This, however, is not intended to suggest that there is any real cleavage between the two problems or that the analysis as developed in the monograph does not also bear, by implication, upon the prediction problem. Any theory of data smoothing must rest ultimately upon some hypothesis concerning the path of the target, and the exact statement of the assumptions to be made is in many ways the most important as well as the most difficult part of the problem. The same assumptions, however, are also involved in the extrapolation to the future position of the target. It is thus impossible to solve the data-smoothing problem without also implying what the general nature of the prediction process will be. For example, the formulation given in Chapter 9 amounts to the assumption that the target path is specified by a set of geometrical parameters corresponding to components of velocity, acceleration, etc. The data-smoothing process centers about the problem of obtaining reliable values for these parameters. To obtain a complete prediction thereafter, it is merely necessary to multiply the parameter values thus obtained by suitable functions of time of flight and add the results to the present position of the target.

The other general remark concerns the tactical criteria used in evaluating the performance of a data-smoothing system. This turns out to be one of the most important aspects of the whole field. It is assumed here that the tactical situation is similar to that of antiaircraft fire against high-altitude bombers in World War II. The defense can be regarded as successful if only a fairly small fraction of the targets engaged are destroyed. On the other hand, the lethal radius of the antiaircraft shell is so small that it is also quite difficult to score a kill. Under these circumstances we are interested only in increasing the number of very well aimed shots.

When we combine these assumptions with the path assumptions described in Chapter 9 we are led to the data-smoothing solution formulated here, in preference to the solution obtained with the statistical approach. On the other hand, we might equally well envisage a situation in which the target contained an atomic bomb or some other very destructive agent, so that it becomes very important to intercept it, while the lethal radius of the antiaircraft missile is correspondingly increased, so that great accuracy is not needed for a kill. In this situation our interest would be focused on the problem of minimizing the probability of making large misses, and the solution furnished by the statistical approach would be approximately the best obtainable.[d]

[d] In fairness to the statistical solution it should be pointed out that it is also the best obtainable, without regard to the lethal radius of the shell, if we replace the path assumptions made in Chapter 9 by a "random phase" assumption. The path assumptions in Chapter 9 are almost at the opposite pole from a random phase assumption, and represent a deliberate overstatement, made in order to illustrate the theoretical situation as clearly as possible.

Chapter 7

GENERAL FORMULATION OF THE DATA-SMOOTHING PROBLEM

One of the principal difficulties in any treatment of data smoothing is that of stating exactly what the problem is and what criteria should be applied in judging when we have a satisfactory solution. It is consequently necessary to embark upon a rather extensive general discussion of the data-smoothing problem before it is possible to consider specific methods of designing data-smoothing structures. This preliminary survey will occupy Chapters 7, 8, and 9. As a first step this chapter will describe two of the general ways in which the data-smoothing problem can be approached mathematically. The formulation of the problem which is finally reached in Chapter 9 is not the one which is most obviously suggested by these approaches. This, however, does not lessen their value in characterizing the problem broadly.

7.1 A PHYSICAL ILLUSTRATION

In an actual fire-control system the data-smoothing problem is usually made fairly specific because of the particular geometry adopted in the computer. It may be helpful to have some particular case in mind as a touchstone in interpreting the general discussion. For this purpose the most appropriate example is furnished by long range land-based antiaircraft fire, since most of the analysis described in this monograph was developed originally for its application to this problem. It is usually assumed in the antiaircraft problem that the target flies in a straight line at constant speed, and in one case at least the computer operates by converting the input data into Cartesian coordinates of target position and differentiating these to find the rates of travel in the several Cartesian directions. These rates form the basis of the extrapolation.

The process is illustrated in Figure 1. The input coordinates are transformed into electrical voltages proportional to x_P, y_P, and z_P, the Cartesian coordinates of present position, in the coordinate converter at the left of the diagram. The extrapolation for x is shown explicitly. It consists essentially in differentiating to find the x component of target velocity, multiplying the derivative by the time of flight t_f and adding the result to x_P to find

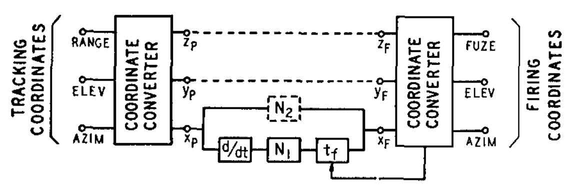


FIGURE 1. Data-smoothing networks in linear prediction circuit.

x_F, the predicted future value of x. A similar procedure fixes y_F and z_F. After the addition of certain ballistic corrections, these three coordinates of future position are transformed into gun aiming orders in the coordinate converter shown at the right of the drawing. This last unit also provides the time of flight required as a multiplier in the extrapolation.

The small irregularities in the input data caused by tracking errors are greatly magnified by the process of differentiation. It is thus necessary to smooth the rates considerably if a reliable extrapolation is to be secured. The data-smoothing network for the x coordinate is represented by N_1 in Figure 1. Since the Cartesian velocity components are theoretically constants if the assumption of a straight line course at constant speed is correct, a data-smoothing network in this computer must be essentially an averaging device which gives an appropriately weighted average of the fluctuating instantaneous rate values fed to it. The problem of "smoothing a constant" is given special attention in Chapter 10. Aside from the particular circuit of Figure 1, we may, of course, be required to smooth a constant whenever the prediction is based upon an assumed geometrical course involving one or more parameters which are isolated in the circuit.

In addition to smoothing the rates we can, if we like, attempt to smooth the irregularities in present position also. A network to accomplish this purpose is indicated by the broken line structure N_2 in Figure 1. Of course, in dealing with the present position we are no longer smoothing a constant, but suitable structures can be obtained by methods described later. However, the effect of tracking errors in the present position circuit is so much less than it is in the rate circuit that N_2 can generally be omitted.

Geometrical assumptions of the sort implied in Figure 1 are helpful in visualizing the problem, and they are of course of critical importance in determining what the final data-smoothing device will be. It is important not to make explicit assumptions of this kind too early in the formal analysis, however, since the meaning of such assumptions is one of the aspects of the general problem which must be investigated. For example, it is apparent that no airplane in fact flies exactly a straight line, nor flies a straight line for an indefinite period. In detail, the solution of the data-smoothing problem depends very largely on how we treat these departures from the idealized straight line path. For the present, consequently, it will be assumed that the input data are presented to the data-smoothing and predicting devices in terms of some generalized coordinates, the nature of which we will not inquire into too closely. A given coordinate might, for example, be a velocity, a radius of curvature, an angle of dive or climb, or any other quantity which would be directly useful in making a prediction, or it might be a simple position coordinate such as an azimuth or an altitude.

The data-smoothing and predicting operation itself is assumed to be performed by linear invariable devices. Aside from the fact that this assumption is, of course, a tremendously simplifying one, it also fits the data-smoothing problem very nicely, as the problem is formulated in this chapter. With other formulations, however, it appears that somewhat better results may be obtainable from variable devices or devices including more or less radical amounts of nonlinearity. These possibilities are discussed briefly in Chapter 14.

7.2 DATA SMOOTHING AND PREDICTION

Figure 1 illustrates a distinction between two possible methods of looking at the data-smoothing problem which it is advisable to establish for future purposes. In describing the x system in Figure 1 we laid emphasis on the particular networks N_1 and N_2. It is clear, however, that the complete x circuit with input x_P and output x_F is a network having overall transmission properties which can be studied. Since t_f will normally vary with time, the network is not, strictly speaking, an invariable one, but the changes of t_f are ordinarily too slow to make this an essential consideration.

When it is necessary to make a distinction between these points of view, a network such as N_1, which is merely an element in the prediction process, will be called a *data-smoothing* structure. An overall circuit, providing data smoothing and prediction in one step, will be called a *data-smoothing and prediction* network, or simply a *prediction* network. Although these points of view have been illustrated for rectangular coordinates, they obviously apply also in many other situations. For example, we might go so far as to apply the overall point of view to a complete circuit from input azimuth, say, to output azimuth.

Both points of view are taken from time to time in the monograph. When possible, however, principal attention has been given to the limited data-smoothing problem. This tends to simplify the discussion, since the limited problem is evidently more concrete than the overall prediction problem. Moreover, it permits us to deal lightly with such questions as the particular choice of coordinates in which the smoothing operations are conducted, since it assumes that the general kinematical framework of prediction has already been decided upon. On the other hand, the overall point of view is more effective in certain situations, and it is the only natural one in the statistical treatment described in the next section.

7.3 DATA SMOOTHING AS A PROBLEM IN TIME SERIES

The most direct and perhaps the most general approach to data smoothing consists in regarding it as a problem in time series. This is the approach used by Wiener in his well-known work.[1] It essentially classifies data smoothing and prediction as a branch of statistics. The input data, in other words, are thought of as constituting a series in time similar to weather records, stock market prices, production statistics, and the like. The well-developed tools of statistics for the interpretation and extrapolation of such series are thus made available for the data-smoothing and prediction problem.

To formulate the problem in these terms, let $f(t)$ represent the true value of one of the coordinates of the target and let $g(t)$ represent the observational error. Then $f(t)$ and $g(t)$ are both time series in the sense just defined. The set of all such functions corresponding to the various possible target courses and tracking errors form an ensemble of time series or a statistical population. One can imagine that a large number of particular functions $f(t)$ and $g(t)$ have been recorded, each with a frequency proportional to its actual frequency of occurrence. Wiener assumes that they are *stationary*, that is, that the statistical properties of the ensemble are independent of the origin of time. This, of course, implies that both functions exist from $t = -\infty$ to $t = +\infty$. We will sometimes find it more convenient to make the assumption that the two functions vanish after some fixed, but sufficiently remote, points on the positive and negative real t axis.[a]

The input signal to the computer is of course $f(t) + g(t)$. If we assume that the coordinate in question represents a position, the quantity we wish to obtain is $f(t + t_f)$, where t_f represents the prediction time. If the coordinate is a rate, we are interested in an average value of $f(t)$ over the prediction interval. This complicates the mathematics somewhat, but does not essentially affect the situation.

[a] This is done for technical mathematical reasons. We shall later have occasion to consider the Fourier transforms of $f(t)$ and $g(t)$, and, to have well-defined transforms, the integrals of the squares of the two functions, from $t = -\infty$ to $t = +\infty$, should be finite. This would not happen under the "stationary" assumption. Wiener avoids the difficulty by introducing what he calls a *generalized harmonic analysis*, but this method is far too complicated to be treated in a brief sketch like the present.

We shall not, of course, be able to predict $f(t + t_f)$ perfectly accurately. Let the predicted value be represented by $f^*(t + t_f)$. In virtue of our assumption that the data-smoothing and prediction circuit is to be a linear invariable network, the relation between $f^*(t + t_f)$ and the total input signal $f(t) + g(t)$ can be written as

$$f^*(t + t_f) = \int_{-\infty}^{0} [f(\sigma) + g(\sigma)]dK(\sigma) \qquad (1)$$

where $dK(\sigma)$ represents the effect of the data-smoothing and prediction circuit. Comparison to equations (2) and (5) of Appendix A shows that K is, in fact, the indicial admittance of this circuit. The particular problem to be solved is of course that of finding a shape for the function $K(\sigma)$ which will make $f^*(t + t_f)$ the best possible estimate of $f(t + t_f)$.

The fact that the upper limit of integration in equation (1) is taken as $\sigma = 0$ is particularly to be noted. It corresponds to the fact that in making a prediction we are entitled to use only the input data which has accumulated up to the prediction instant. This restriction will be conspicuous in the next chapter, where the time-series analysis is completed.

7.4 THE AUTOCORRELATION

The principal statistical tool used in studying equation (1) is the so-called autocorrelation. Under the "stationary" assumption the autocorrelation for $f(t)$ is defined by

$$\phi_1(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t + \tau)f(t)dt\,. \qquad (2)$$

We can obtain a normalized autocorrelation, which is more convenient for some purposes, by dividing by $\phi_1(0)$. This gives

$$\phi_{1N}(\tau) = \frac{\phi_1(\tau)}{\phi_1(0)} = \lim_{T\to\infty} \frac{\int_{-T}^{T} f(t + \tau)f(t)dt}{\int_{-T}^{T} [f(t)]^2\, dt}\,. \qquad (3)$$

If we assume that $f(t)$ in fact vanishes for sufficiently large positive or negative values of t, the limit sign can be disregarded and $\phi_{1N}(\tau)$ becomes simply

$$\phi_{1N}(\tau) = \frac{1}{W_1}\int_{-\infty}^{\infty} f(t+\tau)\, f(t)dt \qquad (4)$$

where $W_1 = \int_{-\infty}^{\infty} [f(t)]^2 dt$ and represents the total "energy" in the time series $f(t)$.

Precisely similar expressions can be set up for the autocorrelation $\phi_2(\tau)$ or $\phi_{2N}(\tau)$ of the observational error function $g(t)$. In a general case we might also have to worry about a possible cross correlation between $f(t)$ and $g(t)$. This would be represented by a cross-correlation function $\phi_{12}(\tau)$, obtained by integrating the product $f(t+\tau)g(t)$. In practical fire control, however, it can be assumed that the correlation between target course and tracking errors is small enough to be neglected.

As a simple example of the calculation of an autocorrelation we may assume that $f(t) = \sin \omega t$. Then

$$\phi_1(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin \omega(t+\tau) \sin \omega t \cdot dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{1}{2}[\cos \omega\tau - \cos(2\omega t + \omega\tau)]dt$$

$$= \frac{1}{2}\cos \omega\tau, \qquad (5)$$

since the term $\cos(2\omega t + \omega\tau)$ will contribute nothing in the limit.

The maximum value of $\phi_1(\tau)$ in (5) is found at $\tau = 0$. This is to be expected, since obviously the correlation between identical values of the function is the best possible. What is exceptional about the present result is the fact that $\phi_1(\tau)$ is not small for all large τ's. This is fundamentally a consequence of the fact that we chose an analytic expression for $f(t)$, so that the relation between two values of the function is completely determinate, no matter how great the difference between their arguments. In a more representative time series, involving a certain amount of statistical uncertainty, we would expect $\phi_1(\tau)$ to approach zero as τ increases, reflecting the increasing importance of statistical dispersion as the time interval becomes greater.

The significance of the autocorrelation function for data smoothing and prediction is obvious without much study. Thus, suppose for simplicity that the observational error $g(t)$ is zero. Then the autocorrelation $\phi_1(\tau)$ is the only one involved. It is a measure of the extent to which the true target path "hangs together" and is thus predictable. For example, in weather forecasting it is a well-known principle that in the absence of any other information it is a reasonably good bet that tomorrow's weather will be like today's but that the reliability of such a prediction diminishes rapidly if we attempt to go beyond two or three days. This would correspond to an autocorrelation function which is fairly large in the neighborhood of $\tau = 0$, but diminishes rapidly to zero thereafter.

In a similar way the autocorrelation of the observational error $g(t)$ represents the extent to which this error hangs together. In this case, however, a high correlation is exactly what we do not want. Thus, if $\phi_2(\tau)$ vanishes rapidly as τ increases from zero, closely neighboring values of g are quite uncorrelated, and we need only average the input data over a short interval in the immediate past in order to have most of the observational errors averaged out. If $\phi_2(\tau)$ is substantial for a much longer range, on the other hand, a much longer averaging period is necessary, with correspondingly greater uncertainties in the value obtained for $f(t)$.

7.5 THE LEAST SQUARES ASSUMPTION

The autocorrelation function does not in itself suffice to determine a time series completely. For example, it is easily seen that the functions $\sin t + \sin 2t$ and $\sin t + \cos 2t$ have the same autocorrelation in spite of the fact that they represent waves of quite different shape. The autocorrelation function, however, has a peculiar importance in the fact that under many circumstances it is the only piece of information about the time series which we need to know.

The significance of the autocorrelation becomes apparent as soon as we investigate the error in prediction. In many mathematical situations involving linear systems it is convenient to deal with the square of the error rather than with the error itself, since a first variation in the error squared expression gives a linear relationship in the quantities of direct interest. We will deal with the square of the error here. If E represents the instantaneous error, $f^*(t + t_f) - f(t + t_f)$, the mean square error over a long period of time is evidently

$$\overline{E}^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f^*(t + t_f) - f(t + t_f)]^2 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f(t + t_f)]^2 \, dt$$

$$- \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} f(t + t_f) f^*(t + t_f) \, dt$$

$$+ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f^*(t + t_f)]^2 \, dt. \tag{6}$$

The first integral in equation (6) can be evaluated immediately. From (2) it is $\phi_1(0)$. To evaluate the second integral replace $f^*(t + t_f)$ by its definition from (1). This gives

$$- \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} f(t + t_f) dt \int_{0}^{\infty} [f(t - \tau) + g(t - \tau)] dK(\tau) = - \lim_{T\to\infty} \frac{1}{T} \int_{0}^{\infty} dK(\tau)$$

$$\int_{-T}^{T} [f(t + t_f) f(t - \tau) + f(t + t_f) g(t - \tau)] \, dt$$

if we reverse the order of integration. Since we assume that f and g are uncorrelated, however, the product $f(t + t_f) g(t - \tau)$ in this expression makes no contribution to the final result, and by replacing the integral of $f(t + t_f)$ $f(t - \tau)$ by its value in terms of ϕ_1 the expression as a whole can be written as

$$- 2 \int_{0}^{\infty} \phi_1(t_f + \tau) \, dK(\tau).$$

The third integral in (6) can be simplified in similar fashion. The final result becomes

$$\overline{E}^2 = \phi_1(0) - 2 \int_{0}^{\infty} \phi_1(t_f + \tau) \, dK(\tau) + \int_{0}^{\infty} dK(\tau) \int_{0}^{\infty} [\phi_1(\tau - \sigma) + \phi_2(\tau - \sigma)] dK(\sigma). \tag{7}$$

The only quantities appearing in equation (7) are the autocorrelations, ϕ_1 and ϕ_2, of the true target path and the observational error, and the function K which specifies the data-smoothing structure. The theoretical problem with which we are confronted is evidently that of choosing K to make the mean square error as small as possible for any given ϕ's. This problem will not be attacked here, although a solution obtained by a somewhat indirect method is presented in the next chapter. The principal reason for deriving equation (7) is to demonstrate the very important fact that *the mean square error depends only upon the two autocorrelations.* No other characteristics of the input data need be considered.

It will be recalled that the mean square criterion was introduced originally on the ground of mathematical convenience. This leaves unsettled the question of how good a measure of performance for a data-smoothing network it actually is. This is a critical question, since upon it depends the validity of the whole approach outlined in this chapter. A priori, the least squares criterion is a dubious one since it gives principal weight to large errors. In fire control we are normally interested only in shots which are close enough to register as hits. If a shot misses it makes little difference whether the miss is large or small. The merits of the least squares criterion are considered in more detail in Chapter 9, where the conclusion is reached that the criterion is probably adequate for many problems but needs to be supplemented or replaced in others, including the special case of heavy antiaircraft fire to which particular attention is given in this monograph. Pending the discussion in Chapter 9, the least squares criterion will be assumed to be a valid one, with the understanding that the analysis is intended primarily for its value in contributing to the general understanding of the data-smoothing problem rather than as a means of fixing the exact proportions of an optimal smoothing network.

7.6 DATA SMOOTHING AS A FILTER PROBLEM

The time-series approach to data smoothing is closely associated with another which at first sight may seem quite different. This second approach is suggested by the procedures used in communication engineering. Here the signals, be they voice, music, television, or what not, are again time series. Instead of dealing with actual signals varying in a more or less irregular and random manner with time, however, it is customary to deal with their equivalent steady-state components on the frequency spectrum.[b]

The analysis of data smoothing can conveniently be approached by supposing that both the true path of the target and the effects of tracking errors are represented, in a similar way, by their frequency spectra. When the situation is presented in this way, however, there is an obvious analogy between the problem of smoothing the data to eliminate or reduce the effect of tracking errors and the problem of separating a signal from interfering noise in communication systems. We may take as an example of the latter the transmission of voice or music by ordinary radio over fairly long distances, so that the effects of static interference are appreciable. In such a system a reasonable separation of the desired signal from the static can be obtained by means of a filter. In a representative situation an appropriate filter might transmit frequencies up to perhaps 2,000 or 3,000 cycles per second,[c] while rejecting higher frequencies.

The choice of any specific cutoff, such as 2,000 or 3,000 c, in the radio system depends upon a compromise between conflicting considerations. Both speech or music and static normally include components of all frequencies which can be heard by the human ear. Thus, suppressing any frequency range below the limits of audibility, at perhaps 10,000 or 20,000 c, will injure the signal to some extent. The intensity of the signal components, however, diminishes rapidly above 2,000 or 3,000 c, while the energy of the static interference is more evenly distributed over the spectrum. Thus, by filtering out the first 2,000 or 3,000 c, we can retain most of the signal while rejecting most of the noise. Naturally, the exact dividing line will depend upon the relative levels of signal and noise power. If the static interference is quite weak, for example, it would be worth while to transmit a considerably wider band in order to retain a more nearly perfect signal. If the static level is extremely high, on the other hand, it would be necessary to transmit a still narrower band at the cost of greater mutilation of the signal.

The separation of the true path of a target from the observed path including tracking errors, as a preliminary to prediction of the future position of the target, presents an approximately analogous situation. Again the spectrum of the "signal" or true path is concentrated principally in a low-frequency band, in most instances, while the energy of tracking errors or "noise" appears principally at considerably higher frequencies. Thus the two can be separated by a low-pass filter. The separation, however, is not complete since some components of the signal spectrum extend into the noise region. Thus the smoothing process must be accompanied by some mutilation of the signal, and the optimum compromise is again attained from a filter which transmits a relatively broad band when the tracking errors are of low intensity and a much narrower band when they are large.

In these terms the most obvious difference between the data-smoothing problem and the static interference problem in the radio system is in the order of magnitude of the frequencies involved. They are roughly 10,000 times smaller in the data-smoothing case. Thus, the typical signal band in a fire-control system may cover a few tenths of a cycle per second, in comparison with a useful band of 2,000 or 3,000 c in a radio system, and the spectrum of tracking errors or noise, with representative tracking devices, includes appreciable components up to perhaps 2 or 3 c, in comparison with a total effective noise band in the radio system extending to the limits of audibility at perhaps 20,000 c.

This analogy between data smoothing and the filtering problems which appear in ordinary communication systems transmitting speech or music must of course not be carried too far. For example, previous experience with communication filters is of no help in fixing in detail the cutoff in attenuation characteristic of the data-smoothing filter, since in communication systems these choices depend on psycho-

[b] The review of communication theory given in Appendix A shows how this equivalence is established by Fourier or Laplace transform methods.

[c] In practice, of course, the filtering would probably take place in the radio-frequency circuits, but it is more convenient here to think of it occurring in the demodulated output.

logical considerations of no relevance in the fire-control problem. Methods of determining the best rules for proportioning a data-smoothing filter, therefore, remain to be determined. We may also notice that, whereas the time-series approach was of the *data-smoothing and prediction* type, the filter approach emphasizes *data smoothing* only. The addition of the prediction function can be expected to change materially the overall characteristics of the circuit. Neither of these remarks, however, robs the filter approach of its value as a simple way of thinking about the problem qualitatively.

7.7 RELATION BETWEEN TIME-SERIES AND FILTER APPROACHES

The time-series and filter methods of looking at data smoothing are related to one another by the fact that the autocorrelation can be computed from the amplitude spectrum, or vice versa, by Fourier transform means. Consider, for example, the Fourier transform of the autocorrelation. If we make use in particular of (4) we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi_{1N}(\tau)e^{-i\omega\tau}d\tau$$
$$=\frac{1}{\sqrt{2\pi}\,W_1}\int_{-\infty}^{\infty} e^{-i\omega\tau}d\tau \int_{-\infty}^{\infty} f(t+\tau)f(t)dt$$
$$=\frac{1}{\sqrt{2\pi}\,W_1}\int_{-\infty}^{\infty} f(t)dt \int_{-\infty}^{\infty} f(t+\tau)\, e^{-i\omega\tau}d\tau$$
$$=\frac{1}{\sqrt{2\pi}\,W_1}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}dt \int_{-\infty}^{\infty} f(t+\tau)e^{-i\omega(t+\tau)}\, d\tau$$
$$=\frac{\sqrt{2\pi}}{W_1}F(\omega)\overline{F}(\omega) \tag{8}$$

where

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t+\tau)\, e^{-i\omega(t+\tau)}\, d\tau\,. \tag{9}$$

$F(\omega)$ is of course the steady-state spectrum of the signal $f(t)$. Equation (8) thus states that the Fourier transform of ϕ_{1N} is equal to a constant times the square of the amplitude of the steady-state spectrum. The amplitude squared spectrum is, however, a measure of the power per cycle. The relation is therefore equivalent to the statement that the autocorrelation and power spectrum are Fourier transforms of each other.

Since we have already established the fact that the mean square error in prediction depends only on the autocorrelation, this analysis enables us to conclude immediately that the mean square error can also be calculated from the power spectra of the signal and noise. It is entirely independent of the phase relations in either signal or noise. The phase characteristics of the data-smoothing network, which operates on the signal after a specific wave shape has been established, is, of course, still of consequence.

7.8 PHYSICAL AND TACTICAL CONSIDERATIONS

Thus far the material which has been presented has been primarily mathematical. It has consisted, in other words, of outlines of general analytical methods which are available for use with the data-smoothing problem. It is also possible to approach the problem in a much more concrete fashion. It is obvious that by giving thought to the details of the physical characteristics of tracking units and targets, and to the tactical situations with which we expect to deal, it should be possible to draw a number of specific conclusions about the problem as a whole. In a general theory of the design and tactical use of fire-control apparatus such an approach might well be a primary one. It is scarcely possible to follow it in detail in the present discussion. The following paragraphs, however, indicate some of the kinds of considerations which can be brought into the problem in this way. It will be seen that they tend to modify the strictly mathematical approach, partly by qualifying to some extent the assumptions made in the mathematics, and partly by tending to give much more emphasis to particular aspects of the problem than would appear in a general analytic outline.

Choice of Coordinates

One of the most obvious omissions in the general analysis thus far is any consideration of the choice of coordinates in which the data smoothing is to take place. So far as either the statistical or filter theory is concerned, the coordinates in the data smoother may represent either the original tracking data or any transformation of them. The fact that there is actually something to be decided here, however, is easily seen from the long-range antiaircraft problem. The input tracking coordinates for antiaircraft would normally be azimuth, elevation, and slant range. If the airplane flies in a straight line roughly overhead, the general shape of the azimuth and the azimuth rate as functions of time are given by the curves in Figure 2. The curves become indefinitely

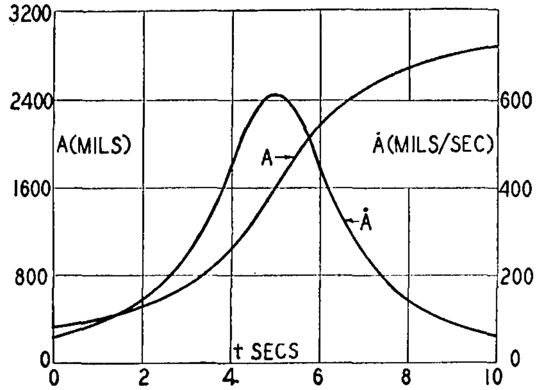


FIGURE 2. Azimuth and azimuth rate for crossing target.

steeper as the target path approaches the zenith, and it will be seen that if the approach is reasonably close, either the azimuth or the azimuth rate must include a very substantial amount of high-frequency energy. Since the possibility of an effective separation between the signal and noise in the filter approach depends upon the assumption that the signal components are of quite low frequency with respect to the noise, the presence of this high-frequency energy is evidently serious.

When the target describes a violently evasive path the signal spectrum must naturally include substantial high-frequency components, whatever the coordinate system may be. The high-frequency components indicated in Figure 2, however, are due to the fact that the target path happens to pass almost over the director and are essentially superimposed upon the high-frequency components which reflect the complexity of the target path itself. It is clear as a matter of principle that an acceptable coordinate system for data smoothing should not introduce frequency components which depend upon such accidental factors as the location and orientation of the coordinate system. The rectangular system mentioned in connection with Figure 1 evidently meets this condition; so also does the "intrinsic" system described in the next section.

PHYSICAL LIMITATIONS OF TARGET OR TRACKER

We may also approach the data-smoothing question by a consideration of the motions which are physically possible either in the target or in the tracking device. In the heavy antiaircraft problem, for example, there are substantial physical limitations on the performance possibilities of present-day aircraft. We can be quite sure that any motion incompatible with these limitations is necessarily a tracking error and can be removed from the incoming data. Naturally, these limitations must appear in the power spectrum of the signal if they affect the mean square error in prediction, so that their existence in no way disputes the mathematical framework we have set up. Consideration of the physical factors which produce them, however, may permit them to be established more easily or in more clear-cut fashion than would be possible from a statistical examination of target records alone.

The limitations on airplane performance can be stated most simply when the motion of the airplane is expressed in so-called intrinsic coordinates. These are the speed of the airplane, its heading, and its angle of dive or climb. The maneuvering possibilities of a conventional airplane in these three directions are quite unequal. By banking sharply it can maneuver violently to the right and left and thus make quick changes in heading. The possibilities of maneuvering up and down, however, are considerably less, particularly for a heavy airplane, where there are usually restrictions on the maximum angle of dive or climb which can be assumed. The possibilities of quickly changing the speed of the airplane, finally, are almost nil. The thrust of an airplane propeller is so small in comparison with the mass of the airplane that only small accelerations are possible.[d]

Thus the optimum filters for the three coordinates should be different. The one for speed can have a very narrow band, since most of the signal energy for this coordinate occurs at very low frequencies. The optimum band for the angle of dive or climb, however, should be larger (unless it turns out that pilots seldom make use of maneuvering possibilities in this direction) and the one for the heading larger still. In this ability to discriminate among the various possible directions of motion the intrinsic coordinate system is evidently an improvement even on the rectangular system.

Settling Time

Another aspect of the data-smoothing problem which has not been given conspicuous attention in the purely mathematical discussion is the fact that in an actual tactical situation questions of elapsed time are of great importance. Engagements usually begin suddenly and last for a comparatively brief period, and it is important to find a data-smoothing scheme which provides adequate firing data as quickly as possible after an engagement starts. A situation essentially similar to the beginning of an engagement may also be presented whenever the target makes a sudden change of course or whenever it is necessary to shift from one target to another in a given attacking body. The time required for a computer to give usable output data after any of these events is its so-called "settling time," and is one of the most important parameters of any data-smoothing system. It is possible to make rough estimates of settling time by indirect means in both the statistical and filter theories of data smoothing, but no explicit consideration of necessary time lapses appears in either theory. Evidently, the fundamental fault lies with the "stationary" assumption.

[d] This ignores the possibility of changing the speed through gravitational forces. Since these possibilities are linked to the angle of dive or climb, however, they can be predicted. This has actually been done in one experimental computer.

Effect of Human Factors

Aside from the conditions on target performance which arise from the physical characteristics of the target itself, there are others which are due to the fact that the target is under the control of a human being with a definite purpose. The language of the statistical and filter methods is broad enough to cover almost any situation. It tends to suggest, however, that the typical target paths with which we deal are the relatively structureless consequences of random physical forces. The intervention of purposive human behavior, on the other hand, tends to give paths which fall into more or less definite patterns. A simple illustration is furnished by the argument which is frequently offered in defense of the straight line assumption in dealing with antiaircraft defense against heavy bombers. It is contended that while the targets may in fact engage in substantial evasive maneuvers during most of their flight, there will always be a substantial period during the bombing run in which they must fly very straight in order to achieve bombing accuracy. On the basis of ordinary probability we would of course expect substantial straight line segments quite infrequently if the course as a whole shows marked dispersion, and the intervention of the human pilot thus provides a higher degree of structure than one would expect in a corresponding situation dominated by purely natural factors.

A broader example is furnished by a comparison of two airplanes, or perhaps more simply of two boats, one of which is under the control of a human operator, while in the other the steering controls are lashed in a neutral position. Both boats, say, may be expected to experience small variations of course due to the random effects of wind and waves upon them. Over a short period of time the observed motions of the two boats should be substantially identical. In the case of the boat with the lashed helm these random variations will tend to accumulate, so that it is possible to make a reasonable prediction of the position of the boat for only a comparatively short distance in the future. In the boat with the human steersman, on the other hand, we may expect corrections to be applied as soon as the random effects become large, so that the boat tends to retain the same general course and it is possible to predict its position hours or even days later from a relatively brief observation.

Neither of these illustrations is inconsistent with the mathematical framework laid down earlier in the chapter, in a purely theoretical sense. For example, the bombing run illustration merely states that because of the presence of the human operator there are definite phase relations in the input signal. As we have seen, such relations can exist without affecting computations based on mean square error. The comparison between the piloted and pilotless boats can be interpreted as the result primarily of differences in the signal power spectrum. In the case of the pilotless boat, for example, the signal occupies a fairly continuous low-frequency band, while in the case of the piloted boat it must be regarded as concentrated very closely around zero frequency, so that it is approximately a line spectrum superimposed on a continuous one. The formal mathematical theory covers also such cases as these.

The point of this discussion, however, is that the mathematical theory, although it is sufficiently general in a formal sense, fails to differentiate between such situations as those just described and the more shapeless sort involving continuous spectra with random phase relations, even if the special features in these situations may be the controlling factors in determining the actual probability of hitting. If we could believe the bombing run hypothesis, for example, and had a sufficiently accurate computer and gun, we could expect to score a hit in every engagement, no matter how large the mean square error might be. More generally, it is probably only the tendency of targets to exhibit "line spectra" which prevents the real probability of a kill, small at best, from becoming microscopic. It is necessary to lay special emphasis on these factors in order to keep the overall fire control picture in perspective.

Criterion of Performance

Last on this list of doubts about the statistical and filter theories, we may mention the least squares criterion of accuracy. This was discussed before, but it is mentioned again as a matter of emphasis, and because of its close relation with the factors we have just discussed. For example, the bombing run illustration obviously represents one situation in which the mean square error is not a good guide to the actual probability of scoring a hit.

Chapter 8

STEADY-STATE ANALYSIS OF DATA SMOOTHING

It was shown in the previous chapter that both the statistical and filter theory ways of looking at the data-smoothing problem lead naturally to an analysis in terms of the power spectra of the signal and noise. The phase relations are not important as long as we accept the mean square error as a criterion of performance. The inadequacies of the mean square criterion will finally force us to abandon the steady-state attack in favor of a direct analysis in terms of the wave shapes of some assumed signals. The steady-state attack is nevertheless a very useful one. This chapter will consequently continue the analysis from this point of view. It will be assumed as heretofore that the heavy antiaircraft problem is the particular subject of interest.

A large part of the discussion hinges upon the conditions which must be satisfied by the external characteristics of an electrical network if it is to be capable of physical realization in any way whatever. These limitations and the characteristics which may be postulated for physical networks are decisive since, in the absence of such restrictions, no limits could be set upon the performance which might be expected from data-smoothing and predicting circuits. The facts about physically realizable networks which we shall find of most use are summarized below, but the reader not familiar with this field is urged to read also the account given in Sections A.9 and A.10, Appendix A.[15a]

The conditions which must be satisfied by physically realizable networks can be stated in either transient or steady-state terms. In transient terms they are expressed most simply by the statement that the response of a physical network to an impulsive force must be zero up to the time the force is applied. Thus the network has no power to predict a purely arbitrary event. That is, it has no way of foreseeing whether or not an impulse is actually going to be applied to it. This characteristic of physical networks is taken as a postulate.

The steady-state limitations on physical networks are expressed in terms of their attenuation and phase characteristics. They may be derived either from the transient specification or from the postulate that a physical network must be stable. There are no important limitations to be placed upon the attenuation and phase characteristics of physical networks as long as we deal with these characteristics separately, but there are very severe limitations on the phase characteristic which can be associated with any given attenuation characteristic or vice versa. In particular, when the attenuation characteristic is prescribed, there is a definite formula for calculating the unique limiting phase characteristic with which it may be associated.[15b] This is the so-called "minimum phase" characteristic because any other physical network having the postulated attenuation characteristic must have as great or greater phase shift at every frequency. As we shall see later, this greater phase characteristic would correspond to longer lags in obtaining usable data, so that the minimum phase characteristic is the optimum for a data-smoothing network. The minimum phase characteristic has the additional important property that not only does it specify the transfer admittance of a physical network, but the reciprocal of that transfer admittance can also be realized by a physical structure.[a]

In addition to this principal formula for the relation between attenuation and phase there are a number of subsidiary expressions for special aspects of the problem. One in particular, relating the attenuation to the behavior of the phase characteristic in the neighborhood of zero frequency, is used extensively in this chapter.

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[a] In limiting cases, such as may be found when the transfer admittance contains zeros or poles exactly on the real frequency axis, the "physical structure" may require such constituents as ideally nondissipative reactances, perfect amplifiers with unlimited gain, etc. This, however, is of no consequence for the present general discussion.

8.1 THE SIGNAL SPECTRUM

It is natural to begin with a discussion of the spectrum of a typical target path. Unfortunately no data on the spectra of actual measured airplane paths exist, and the theoretical assumptions which may be made about paths of airplane targets are best discussed in the next chapter. This section consequently will be confined to rather general observations about the problem. It will be convenient to assume for definiteness that the quantities to be smoothed are the velocity components in Cartesian coordinates.

The simplest point of departure is furnished by the conventional assumption that the target flies in a straight line at constant speed. If we could construe this assumption literally, it would mean that the velocity spectrum in rectangular coordinates would reduce to a single line at zero frequency. In practice, of course, the spectrum is not so simple. Even in the absence of deliberate maneuvering, the target will fly a slightly curved path because of "wander." Moreover, even if the target could fly exactly straight, the single line spectrum would apply only to a straight course indefinitely continued. The spectrum becomes more complicated if we consider the fact that tracking must have begun at some finite time in the past, or that the target may presumably change occasionally from one straight line course to another.

As a result of both these causes, the actual signal spectrum must be regarded as occupying a band bordering on zero frequency. The distribution of energy in detail will, of course, depend on particular circumstances. The band has no very well defined upper limit, but in most cases the great bulk, at least, of the energy should be below, say, one-fourth or one-fifth of a cycle per second. For example, the natural periods of a heavy airplane, which one would expect to be correlated with wander, are below this limit.[9] This limit is also sufficient to include most of the energy resulting from changes in course occurring as frequently as every ten or twenty seconds.

In general, it is to be supposed that the signal spectrum varies as ω^{-n}, where n may be 1, 2, 3, depending on the frequency range. This follows from general considerations of the limitations of airplane performance. Thus, if we suppose that the velocity changes discontinuously from time to time, it follows from general Fourier principles that the amplitude must vary as ω^{-1}. This is presumably a fair representation of the actual signal spectrum at low frequencies. At moderate frequencies, however, we must take account of the fact that the velocity can actually be changed rapidly but not discontinuously, and we consequently assume that the amplitude begins to vary as ω^{-2}. Finally, at frequencies of the order of perhaps one cycle per second one must take account of the fact that the airplane must bank in order to turn. Since it takes some time to roll into the bank, even the acceleration in the lateral direction cannot be discontinuous, and consequently the amplitude must begin to vary as ω^{-3}. The application of such successive limiting factors in constructing a complete spectrum is described in more detail in Section A.8 of Appendix A.

One other general condition of the same kind can be mentioned. It can be shown[1a] that the integral from zero to infinity of $\log H/1 + \omega^{-2}$, where H is the power spectrum, is very important in determining the properties of a time series. More explicitly, the integral converges if the series is essentially statistical, so that we cannot foretell the future from the past with absolute certainty. This of course is the case with an actual signal spectrum in a fire-control problem. It implies two consequences; first, that H cannot be zero over any finite band; and second, that in the neighborhood of infinite frequency H diminishes slowly enough so that $|\log H|/\omega \rightarrow 0$.

8.2 THE NOISE SPECTRUM

The spectrum of tracking errors depends largely upon the particular sort of tracking equipment involved. Broadly speaking, optical tracking equipment (at least that of the present or recent past) tends to produce tracking errors not only of small amplitude, but also of low frequency, so that they are hard to separate from the signal spectrum. Radar equipment, of the present time, produces higher-frequency errors. Relatively high-frequency errors are particularly likely to be found in very stiff automatic tracking radars.

A number of examples of spectra of tracking errors are shown in Figures 1, 2, and 3. The spectra are given directly in terms of range and angle errors. To make them comparable with the velocity spectra described previously it would be necessary to multiply all amplitudes by ω. In addition, it would of course also be necessary to multiply the angle rates by some suitable range in order to compare them directly with the yards-per-second rates we have otherwise considered.

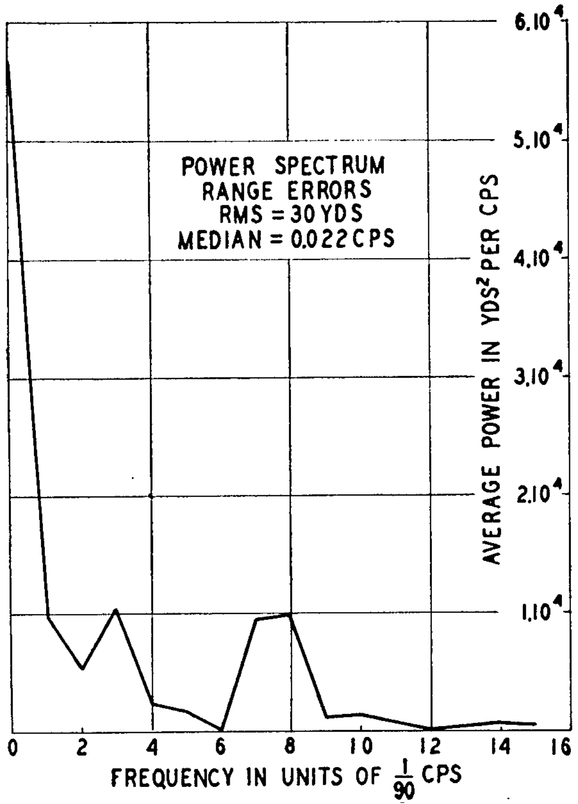


FIGURE 1. Power spectrum of range errors of experimental radar.

After multiplication by ω, the radar spectra appear to be about flat up to perhaps one cycle. Beyond that point they no doubt drop off slowly, although the accuracy of the data is not sufficient to permit the situation to be stated very exactly.

8.3 RANDOM NOISE FUNCTIONS

The properties of the signal and noise as we shall assume them here can be conveniently expressed by reference to the theory of so-called "random noise" functions.[b] A random noise can be defined as a function which has a definite amplitude spectrum but completely random phase characteristics. The theory of such functions is well developed because of their frequent occurrence in physics. It is probable that neither our noise functions nor our signal functions are, strictly speaking, random noise according to this definition. Thus, there are probably certain definite phase relations in our noise functions because of the physical characteristics of tracking devices. There is no evidence, however, that any such relations are important enough to be significant in the data-smoothing problem, so that we are fully justified in identifying them with random noise functions as defined above. The phase relations in the signal are by no means random. As long as we consider only the mean square error, however, this factor is immaterial, and we can replace the actual signal by a random noise function with the same power spectrum for purposes of analysis.

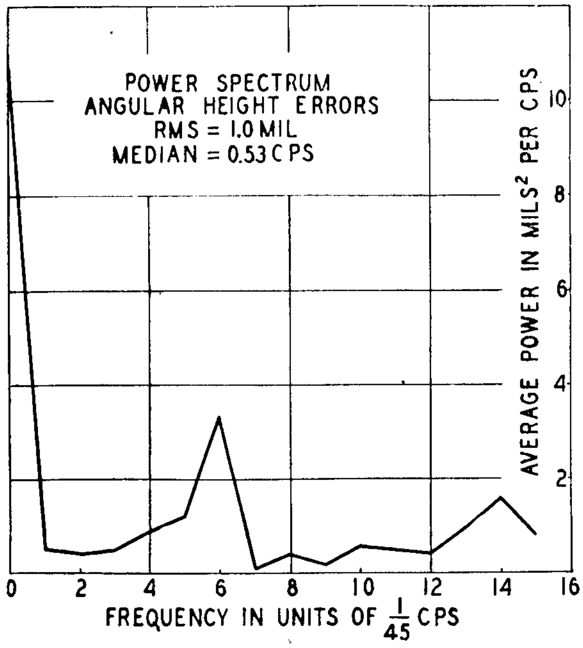


FIGURE 2. Power spectrum of angular height errors of experimental radar.

The most familiar example of a random noise function is furnished by the thermal

[b] The fact that we also refer to tracking errors as "noise" is, of course, merely a coincidence.

voltage across a resistance R. This is a random noise whose spectrum is constant up to very high frequencies with the value $P = 4kTR$ (k is Boltzmann's constant and T the absolute temperature). A second example is black body

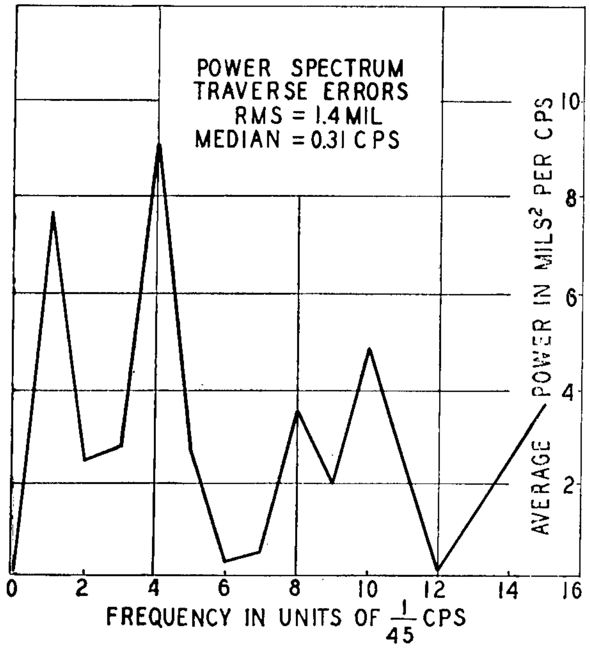


FIGURE 3. Power spectrum of traverse angular errors of experimental radar.

radiation. If there is black body radiation in a space, the electric (or magnetic) field intensity at a point is a random noise function with spectrum

$$P(f) = \frac{8\pi f^3}{C^3} \frac{1}{e^{hf/kt} - 1}$$

according to Planck's law. Random noise functions also occur in the Schottky effect, in Brownian motion, and in diffusion and heat flow problems.

For purposes of analysis, a random noise function can be thought of as a function made up of a large number of sinusoidal components, which are very closely spaced in frequency and whose phases are completely random.[21-23a] Thus a random noise can be represented as

$$\sum_{n=1}^{N} a_n \cos(\omega_n t + \phi_n)$$

where $\omega_n = n\Delta f$, Δf being the frequency difference between adjacent components. The phase angles ϕ_n are random variables which are independent with a uniform probability distribution from 0 to 2π. As Δf decreases the functions in this ensemble approach, in a certain sense, a limiting ensemble, providing the amplitudes a_n are adjusted properly. What is desired is to have the total power in the neighborhood of each frequency approach a certain limit $P(f)$, the power spectrum at that frequency. To do this we make

$$a_n^2 = 2\pi P(f)\Delta f.$$

In the limiting ensemble the total power within a small frequency range Δf is then $P(f)\Delta f$. The function $P(\omega)$ completely describes the random noise ensemble from the statistical point of view.

A particularly important special case is that of a random noise with a constant power spectrum. This is often called "flat" or "white" noise. True constancy out to infinite frequencies is of course impossible since it would imply an infinite total power in the function. The idea is, however, still useful and can be approximated, as with resistance noise, by having a spectrum which is constant out to such high frequencies that behavior beyond this point is of no importance to the problem. We may conveniently think of flat random noise as being made up of a succession of weak impulses occurring frequently but at random times with respect to one another. This results from the fact that a Fourier analysis of a single impulse gives a flat spectrum, and the random occurrence of many of them produces a random set of phases. In a physical problem, such as resistance noise or Brownian motion, these impulses might correspond to the effects of individual small particles. Such a situation is of course completely chaotic. If the impulses are large and occur relatively infrequently, the power spectrum is still flat, though the function is no longer a random noise function as defined here. This conception, which corresponds to a physical situation including definite causative elements, will be revived later under the name of the elementary pulse method of analysis.

Random noise functions have a number of interesting characteristics. For example, they have the "ergodic property." This means that averaging a statistic along the length of a particular random function gives the same results as averaging the same statistic over an ensemble of functions having the same power spectrum. Each function is typical of the ensemble. To be more precise one must admit exceptions, but the probability of an exception is zero. For example, if we determine the fraction of time a given random function $f(t)$ has a value greater than some constant A, it will be equal to the fraction of all functions in the ensemble which are greater than A at $t = 0$ (with probability 1).

A second characteristic of random noise functions is the fact that they frequently lead to Gaussian or normal law distributions. For example, the amplitudes of a random noise function are distributed about zero in accordance with the normal error law. Likewise, the amplitudes for two points spaced a given distance apart form a two-dimensional normal error law distribution when we consider all possible positions of the first point. It is apparent that if the signal and noise are actually random functions the mean square error is as good a criterion of performance as any other, since it completely fixes the distribution in a normal law case.

A final property of random noise functions is the fact that if a random noise is passed through a filter the output is still a random noise. If the power spectrum of the noise is $P(\omega)$ and the transfer characteristic of the filter is $Y(i\omega)$, the output spectrum is $P(\omega)|Y(i\omega)|^2$. In particular, if we take the derivative of a random noise with spectrum $P(\omega)$ we obtain one with spectrum $\omega^2 P(\omega)$.

This last property of random noise functions suggests a method of representing them which we shall find useful in the future. The method is represented by Figure 4. It consists of a

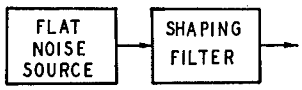


FIGURE 4. Circuit representation of random functions.

source of flat noise followed by a shaping filter to give the desired power spectrum. We can easily assign to the filter the characteristics of a physically realizable structure by making use of the relations between attenuation and phase mentioned earlier in the chapter. It is merely necessary to convert the desired power spectrum into a specification of the attenuation characteristic of the filter and then use the loss-phase formula to compute the corresponding phase shift. It will be assumed that this procedure has been followed when we make use of this circuit at a later point.

The method of representing random functions shown by Figure 4 illustrates graphically the basis of the prediction schemes described thus far. The flat noise is of course absolutely unpredictable. The history of the function up to any given instant gives no indication of its value even a microsecond later. The filter, however, forces the output current to have a certain structure on which a prediction may be based. For example, if the filter will pass only very low frequencies it is clear that the output can change very little in a microsecond.

8.4 THEORETICAL PROPORTIONS FOR A DATA-SMOOTHING FILTER

The signal and noise spectra furnish the raw material from which a suitable data-smoothing filter can be deduced. We have still to determine, however, the exact rule for choosing the cutoff and attenuation characteristic of the filter from these spectra. It is clear that previous experience with signal-to-noise problems in systems transmitting voice or music is no help, since the filter proportions here depend upon psychological considerations of no relevance to the fire-control problem. For example, the interfering effect of a small amount of noise is much greater than one might expect from energy considerations, especially in intervals of low message level, and it is consequently worth while to maintain a relatively high level of attenuation in the noise band. Conversely, the breadth of the band required for the message depends as much on the ability of the ear to reconstruct a complete signal from an incomplete one as it does upon the actual signal power spectrum.

In the data-smoothing case a suitable criterion, dependent upon more physical considerations, can be obtained by minimizing the rms error at the filter output. This criterion is easily developed from the power spectrum approach, and in a sense it is, of course, the only possible one as long as we follow the methods developed thus far.

A very general theory for the minimization of the rms error of the filter output has been developed by Wiener.[1] Since the power spectrum approach is not the one we shall eventually follow, however, it is not necessary to give this analysis in detail. The nature of the relationships can be seen from an elementary computation. Thus in Figure 5 let OA be a unit

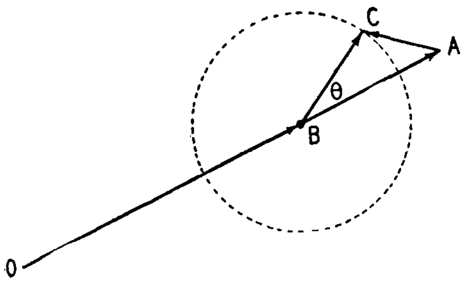


FIGURE 5. Vector relation between input and output of data-smoothing network.

vector representing the signal component at some particular frequency. Let the amplitude ratio between the input and output of the data-smoothing filter be x, and let it be assumed that the system is phase distortionless. This can always be accomplished, at the cost of lag, by phase equalization. Then the actual signal output can be represented by OB, where $OB/OA = x$. Let the ratio of noise power to signal power at this frequency be k^2. Then the output noise can be represented by the vector BC, at some arbitrary phase angle θ, where $BC/OA = kx$.

The error in the output of the data-smoothing filter is evidently represented by the vector AC. We have

$$\begin{aligned}(AC)^2 &= (OA)^2[(1 - x - kx\cos\theta)^2 + (kx\sin\theta)^2] \\ &= (OA)^2[(1 - x^2) - 2kx(1 - x)\cos\theta + k^2x^2].\end{aligned}$$

Since θ is random the cross-product term involving $\cos\theta$ disappears on the average. (More generally, it disappears as long as the noise and signal are uncorrelated, whether or not their relative phases are entirely random.) This leaves the mean square error as

$$(AC)^2_{\text{mean}} = (OA)^2[1 - 2x + (1 + k^2)x^2]. \quad (1)$$

The mean square error is a minimum if

$$x = \frac{1}{1 + k^2} = \frac{P_S}{P_N + P_S}$$

where P_S and P_N are, respectively, the signal and noise power at this frequency. Upon substituting this result in equation (1) and remembering that $(OA)^2 = P_S$, we find that the minimum mean square error is

$$(AC)^2_{\text{mean min}} = \frac{P_N P_S}{P_N + P_S}. \quad (2)$$

Equation (2) evidently represents the sought-for rule for the filter transmission characteristic. It is illustrated in Figure 6, where P_N

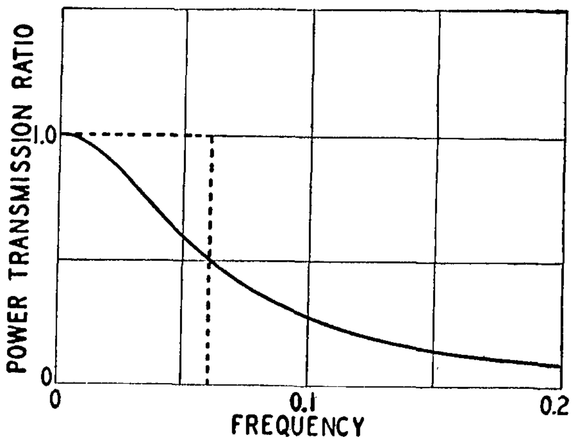


FIGURE 6. Optimum transmission characteristic for data smoothing assuming signals with random noise characteristics.

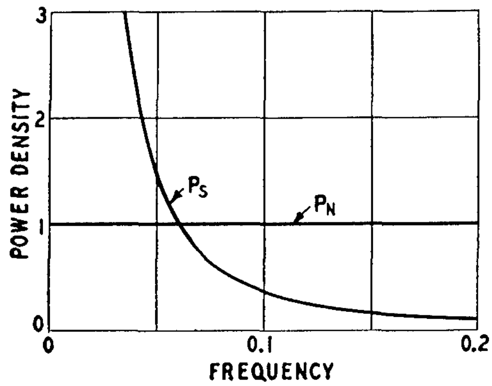


FIGURE 7. Signal and noise power spectra assumed in Figure 6.

and P_S have been chosen respectively as the flat curve and the $1/\omega^2$ curve in Figure 7. In comparison with the characteristics of typical filters in communication systems it is quite rounded with a relatively slowly falling amplitude characteristic. More important than the detailed rule for the transmission characteristic, however, is the conclusion that the shape of the characteristic is not very critical. There is very little loss in replacing the actual curve in Figure 6, by any other similar characteristic. For example, we might validate the assumption of zero phase distortion by making use of the curve which automatically gives a linear phase shift.[15c]

A more extreme illustration is furnished by the infinitely selective filter characteristic, with perfect transmission in the range in which the signal power is greater than the noise power, and zero transmission elsewhere, indicated by the broken lines in Figure 6.

It follows from equation (1) that in the neighborhood of the cutoff point ω_0 the mean square error for this filter is twice that of the optimum structure. In most frequency ranges, however, the penalty is far less than this. Since even a two-to-one change in the mean square error would produce no tremendous improvement in the effectiveness of fire, it is clear that the result to which we are led by this method of attack is by no means critical.

8.5 LAGS IN DATA-SMOOTHING FILTERS

The analysis just concluded has been directed at the amplitude characteristics of a data-smoothing filter. By virtue of the relations between the amplitude and phase characteristics of physical networks mentioned earlier in the chapter, however, the analysis permits us to give at least a partial description also of the phase characteristics of the filters. This is an important consideration because it bears upon the question of time delays in data-smoothing systems which was mentioned in Chapter 7.

The general nature of the relationship in simple cases is illustrated by Figures 8 and 9.

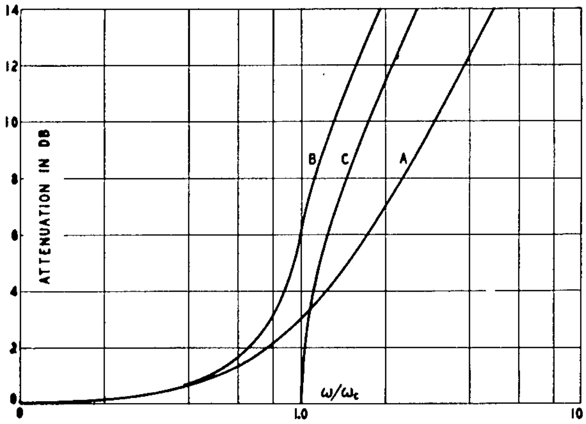


FIGURE 8. Some filter attenuation characteristics.

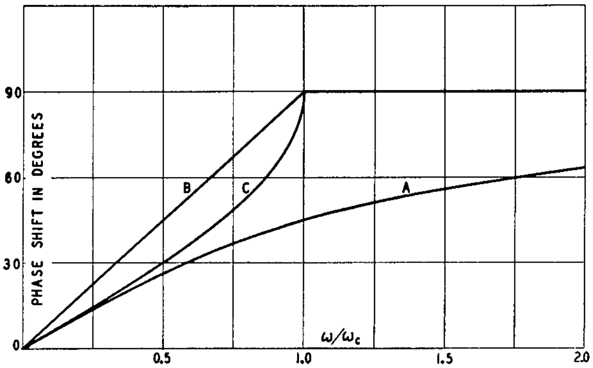


FIGURE 9. Corresponding minimum phase characteristics.

Figure 8 shows a series of rising attenuation characteristics equivalent to rather unselective falling amplitude characteristics of the general type shown by the principal curve in Figure 6. Figure 9 shows the corresponding phase characteristics computed on a minimum phase shift basis. In Figure 8 the central attenuation characteristic B has been so chosen that the corresponding phase characteristic in Figure 9 is exactly a straight line at low frequencies, where the transmitted amplitudes are appreciable. Curves A and C in the two drawings show slightly different cases, but it is clear from the figures that the tendency of the phase characteristics to approximate linearity is still marked.

In communication engineering a phase characteristic proportional to frequency is interpreted as indicating a delay in seconds equal to the slope $dB/d\omega$ of the phase characteristic. This relation is illustrated most simply by an ideal line. The ideal line has zero attenuation combined with a phase shift which is proportional to frequency and which at any given frequency is also proportional to the length of the line in question. If we apply any arbitrary wave to the line it is propagated down the line with a definite velocity and unchanged wave form. The time required for the wave to reach any point on the line is equal to the slope of the phase characteristic to that point.

In a structure like a filter, which has an attenuation characteristic varying with frequency, it is of course no longer possible to transmit an arbitrarily impressed wave without change in wave shape. Even if the applied wave is merely a suddenly applied d-c voltage or single frequency sinusoid, there is a transient period before the response approximates its final value. In structures having a substantially linear phase characteristic over any frequency range in which they exhibit an appreciable amplitude response, however, this total transient characteristic falls naturally into two parts. The first is a waiting period equal to the slope of the phase characteristic, during which the response is very small, whereas the second is a true transient period in which the response is substantial but does not resemble the final steady-state response. This is illustrated by Figure 10 which shows the voltage at the fifth

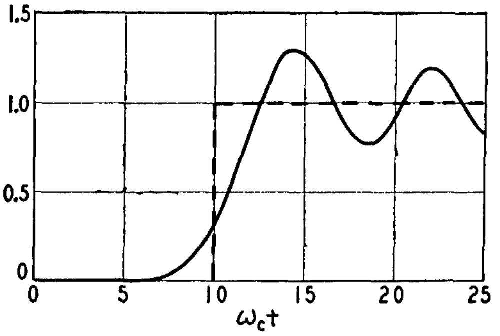


FIGURE 10. Voltage at fifth section of conventional low-pass filter in response to unit d-c voltage.

section of a conventional low-pass filter in response to a d-c voltage applied at zero time at the input terminals.[19] The end of the waiting period, as deduced from the slope of the phase characteristic, is indicated by the broken line.

Delays of the sort just illustrated must be expected in a data-smoothing filter whenever the nature of the signal is changed. This happens at the beginning of tracking, in changing from one target to another, or even in following a single target when the target makes an abrupt change in course. Since usable data in a fire-control system must be quite accurate, the delay to be allowed for must include both the initial waiting period and the subsequent transient period until the transient ripples have almost vanished. A considerable part of the art of designing data-smoothing networks consists in controlling the design so that these final transient ripples decay relatively rapidly. We are not yet ready to discuss this problem. It will turn out, however, that the minimum interval which can be assigned to the "true transient" period is about equal to that which must be allowed for the initial waiting period.[c] Thus the slope of the phase characteristic can be used as an index of the lags which must be expected in data smoothing merely by doubling the delay to which the slope would normally be said to correspond.

When we use the phase slope as an index of delay it becomes immediately apparent that lags are the necessary consequence of smoothing in physical circuits. This is easily seen by reference to the relations which must exist between attenuation and phase characteristics in physical structures. An example is provided by the formula[15d]

$$\left(\frac{dB}{d\omega}\right)_{\omega=0} = \frac{2}{\pi}\int_0^\infty \frac{A - A_0}{\omega^2}\,d\omega = \frac{2}{\pi}\int_0^\infty (A - A_0)\,d\left(\frac{1}{\omega}\right) \tag{3}$$

where A is attenuation, A_0 is the attenuation at zero frequency, and B is phase shift. In other words, the delay (measured by the slope of the phase characteristic at zero frequency) is proportional to the integral of the attenuation on an inverse frequency scale when the attenuation at zero frequency is taken as the reference. The equation thus states that the system will exhibit a lagging response as long as there is a net high-frequency attenuation. As a numerical illustration, let it be supposed that A is zero below $\omega = 1$. This corresponds to the estimate made earlier in the chapter that the input signal components in antiaircraft work lie roughly in the band below about 0.1 or 0.2 cycle per second. Let it be supposed also that A at higher frequencies is equal to 3 népers, corresponding to an average amplitude reduction of about 20

[c] This is not intended to imply that the distinction between the initial waiting period and the "true transient" period is quite as sharp as it is in Figure 10. The selectivity in a data-smoothing filter is usually not great enough to justify the assumption that components beyond the linear phase region are of negligible importance.

to 1. Then $dB/d\omega$ at the origin is given from equation (3) as $6/\pi$ seconds, and in accordance with the rule just enunciated the minimum delay to be expected from such a structure in a data-smoothing application would consequently be $12/\pi$ seconds.

Aside from such specific quantitative relations equation (3) is useful as a basis for a number of important qualitative conclusions. One, for example, is the fact that although a lag is a necessary concomitant of any system showing a high-frequency attenuation, the amount of the lag depends greatly upon the portion of the frequency spectrum in which the attenuation is found. Since the integral is taken on an inverse frequency scale, a small attenuation at low frequencies is much more important than a considerably greater attenuation further out in the spectrum. This points to the desirability of designing tracking instruments which generate principally high-frequency noise, even if the amplitude of the noise is somewhat increased thereby. We may also notice that since the attenuation is a logarithmic function of amplitude an initial moderate reduction in the amplitude of disturbing noise may be much less expensive in lag than subsequent attempts at further reduction. For example, an amplitude reduction from 100 to 10 per cent over a given portion of the frequency spectrum produces no more lag than a subsequent reduction from 10 to 1 per cent.

8.6 WIENER'S PREDICTION THEORY—ZERO NOISE CASE

In Chapter 7 we distinguished between what we called the simple data-smoothing problem and the data-smoothing and prediction problem. The simple problem, with which this report is chiefly concerned, is the one which has been given principal attention thus far. On account of its broad interest, however, it seems worth while to include also a brief statement of Wiener's solution of the general problem. The method of development used here is intuitive and nonrigorous in comparison with Wiener's own development, but it permits the principal relations to be established by very elementary means.

It is convenient to consider first the zero noise case. The past history of the signal, then, is known perfectly, and the existence of a prediction problem depends entirely upon the fact that since the signal is assumed to be statistical in character, its future is not completely determined from its past. The situation can be thought of in the terms suggested by Figure 11. The actual signal output appears at

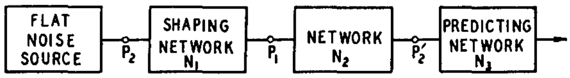


FIGURE 11. Schematic representation of Wiener's prediction theory when there is no noise.

P_1. In accordance with the discussion earlier in the chapter, we imagine this signal to be generated by passing flat noise through the shaping network N_1. The transfer admittance $Y_1(i\omega)$ of N_1 is determined from the power spectrum of the signal by the procedure outlined earlier and is a minimum phase shift characteristic. It will be recalled that minimum phase shift transfer admittances have the important property that their reciprocals are also the transfer admittances of physically realizable networks.

From Y_1 we can readily compute the transient response characteristic of N_1. We shall assume for illustrative purposes that the impulsive admittance of N_1 takes the special shape shown by Figure 12.

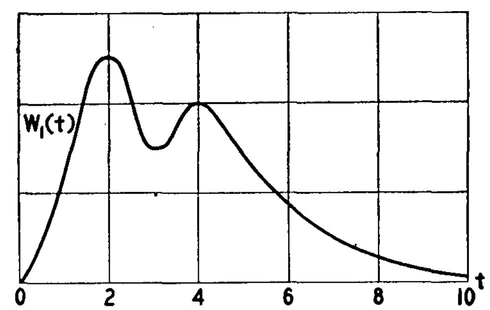


FIGURE 12. Assumed impulsive admittance of shaping filter.

The flat noise is thought of as consisting of a large number of elementary impulses with random amplitudes and occurring at random times. For the purposes of this analysis, however, it is sufficient to consider only the three unit impulses shown in Figure 13. Impulse B is supposed to occur at the instant at which the prediction is to be made, A occurs two seconds in the past, and C, one second in the future. The response of N_1 to these three impulses will evidently be three curves of the sort given by Figure 12, suitably displaced in time as shown by Figure 14.

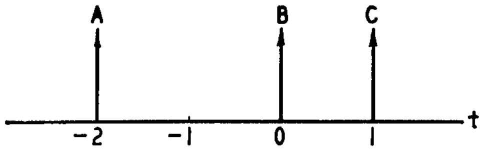


FIGURE 13. Impulses giving rise to applied signal through shaping filter.

The desired output of the predicting network is the curve of Figure 14 advanced by the prediction time, which we can assume, for illustration, to be two seconds. It may be assumed

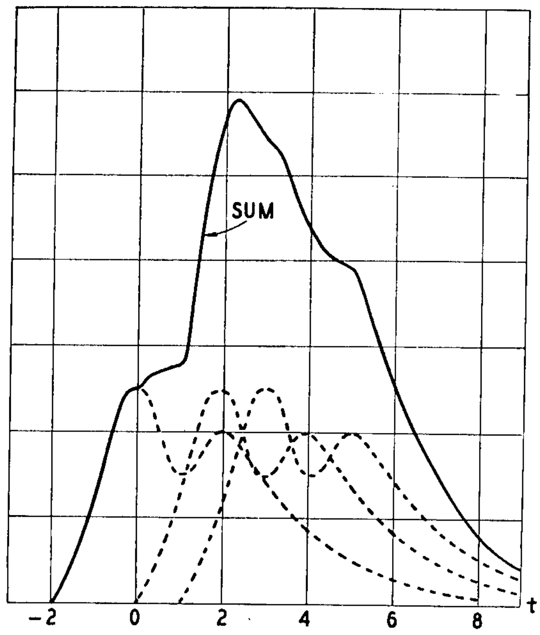


FIGURE 14. Applied signal at P_1.

for the sake of preliminary analysis that the input of the predicting network is the three original impulses of Figure 13. The terminal P_2 at which they are supposed to appear is of course a purely fictitious one and is not accessible to us physically. We can, however, construct the equivalent terminal P'_2 by imposing the actual signal from terminal P_1 on the network N_2, whose transfer admittance is the reciprocal of that of N_1.

Let the predicting network connected to terminal P'_2 be represented by N_3. Obviously a perfect prediction would be secured if N_3 could be assigned the impulsive admittance shown in Figure 15, that is, an impulsive admittance

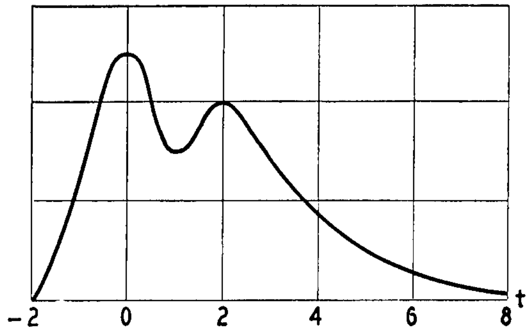


FIGURE 15. Ideal impulsive admittance of prediction network N_3 in Figure 11.

equal to the impulsive admittance of the original network but moved forward by the 2-second prediction time. Then all the constituent curves and the sum curve in Figure 14 would similarly be moved forward. Of course we cannot assign N_3 an impulsive admittance which is different from zero at negative times without postulating a nonphysical network. It is, however, perfectly possible to define N_3 from the portion of the impulsive admittance characteristic at positive times, with the remainder set equal to zero. This gives an impulsive admittance of the type shown by Figure 16. When energized by the three unitary impulses, it gives the result shown in Figure 17. The contributions of impulses A and B are not affected by the absence of a negative time portion of the impulsive admittance, but the contribution of impulse C is lost.

To formulate a physical prediction network

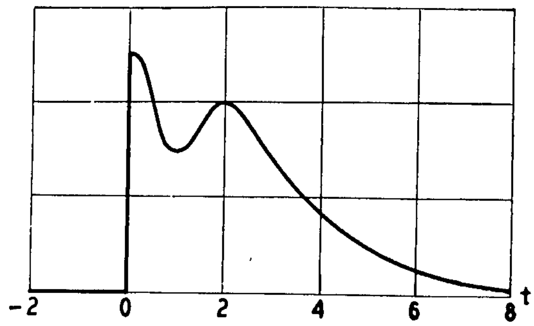


FIGURE 16. Realizable portion of required impulsive admittance.

we have merely to find by conventional methods the steady-state admittance Y_3 corresponding to the impulsive admittance of Figure 16. The two networks N_2 and N_3 may then be

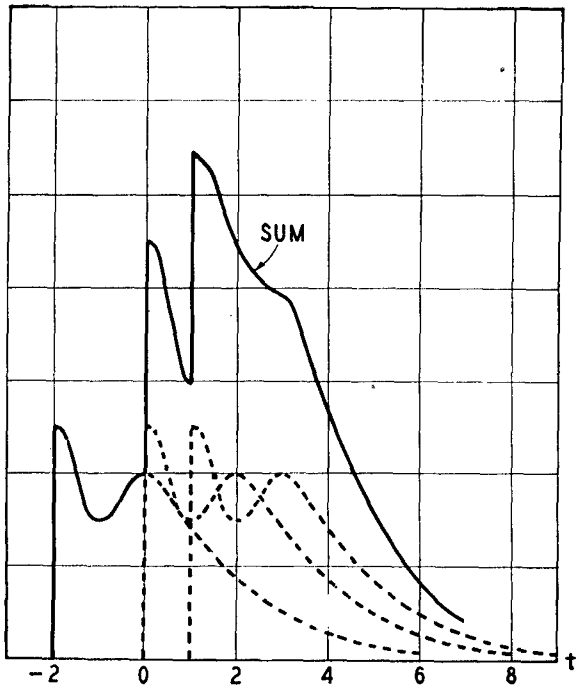


FIGURE 17. Response of realizable prediction network.

combined to give a single structure with the transfer admittance $Y_3Y_2 = Y_3/Y_1$ which will give the complete prediction when energized by the actual signal.

The mean square error in prediction is easily determined from the fact that the contributions of all impulses of the sort represented by C, occurring in the prediction interval, are lost. Since impulses in the flat noise source occur at random times the mean square error is proportional to $\int_0^\alpha W^2(\tau)\,d\tau$, where α is the prediction time and W is the impulsive admittance of Figure 16. Since the flat noise impulses occurring after the time at which the prediction is made are surely unpredictable, it is clear that this error is the least we could expect any physical prediction network to have.

8.7 WIENER'S THEORY—GENERAL CASE

When the input data includes noise as well as the signal it is natural to think of the situation in the manner shown by Figure 18. The first source of flat noise, together with the shaping network N_a, is the combination we have already used to represent the signal in the noise-free

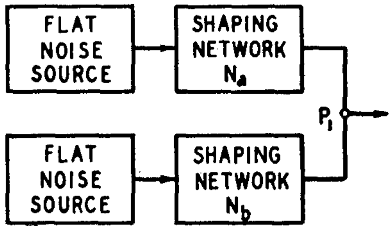


FIGURE 18. Circuit representation of random functions representing signal and noise.

case. The addition of noise is represented by the second independent source of flat noise with its associated shaping network N_b. They combine to give the total input measured at P_1.

This diagram emphasizes the fact that we think of the noise and signal as originating from different physical sources. By postulate, however, we are not able to separate the sources experimentally. So far as any observed result is concerned, consequently, we may as well deal with the simplified structure shown in Figure 19 which contains a single source of

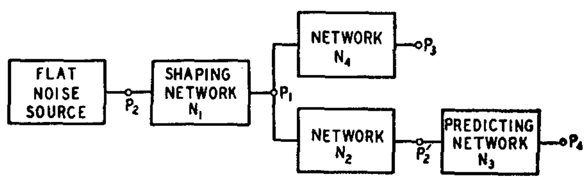


FIGURE 19. Schematic representation of Wiener's prediction theory when there is noise.

flat noise and a single shaping network. The transfer admittance of the shaping network N_1 is determined by adding the power spectra of signal and noise, converting the result to an amplitude characteristic, and computing the corresponding minimum phase according to the methods already used for the noise-free case.[d]

Although we cannot separate the signal from

[d] Note that the shaping network thus obtained is not the same as the one we would secure by adding the transfer admittances of N_a and N_b in Figure 18 directly. In order to realize the same total power at P_1 in each case, it is necessary to begin by adding the powers rather than the amplitude characteristics associated with the two paths.

the noise completely, we saw earlier that the mean square difference between the total input and the signal is minimized if we multiply the amplitude of the input at each frequency by the ratio of the signal power to the sum of the signal and noise powers. A fictitious filter having the prescribed amplitude characteristic is represented by N_4 in Figure 19. We assigned N_4 a zero phase characteristic so that there may be no lag in producing the result at P_3. Thus the output at P_3 at any instant represents the best conceivable estimate (in the least squares sense) of the signal at that instant. The assumption of zero phase, of course, makes N_4 nonphysical, since it must have at least the minimum phase characteristic associated with its prescribed amplitude characteristic. This, however, is not an objection here since the structure is introduced purely for purposes of analysis.

The situation is now reduced to a form in which it is substantially equivalent to the one appearing in the zero-noise case. We assume a series of random impulses at P_2 which would produce responses at P_3. The problem is that of advancing the response to each impulse so that the same result appears α seconds earlier at terminal P_4. The solution is represented by networks N_2 and N_3, which discharge functions similar to those of the correspondingly labeled networks in Figure 11. Thus, the network N_2 is the reciprocal of N_1 and is provided to make terminal P'_2 equivalent to P_2 as a source of impulses. Network N_3 is defined by an impulsive admittance obtained from the impulsive admittance between P_2 and P_3 by advancing the latter characteristic α units in time and then discarding the portion at negative time.

In this procedure there is only one point at which the situation differs from that without noise. In the noise-free case, the original impulsive admittance which we wished to advance in time was identically zero at negative times. In order to secure a physically realizable result, we needed only to discard the portion of the impulsive admittance between $t = 0$ and $t = \alpha$. In the present situation, on the other hand, the impulsive admittance is taken from a path including the nonphysical network N_4. Thus the admittance may be expected to take such form as that shown in Figure 20, with nonzero amplitudes at both negative and positive times, and in order to secure a physical final network it is necessary to discard everything to the left of the line α.

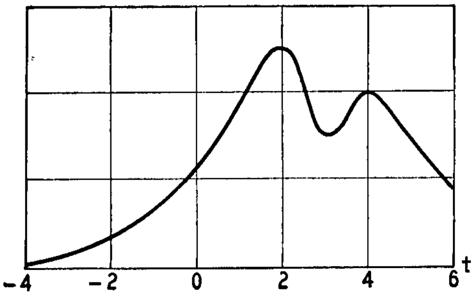


FIGURE 20. Typical impulsive admittance of best smoothing network N_4 in Figure 19.

This difference in the impulsive admittance characteristics has two consequences. The first is the fact that since the uncertainty of the prediction is measured by the amount of impulsive admittance which must be discarded, it is evidently greater in the present case where we are discarding much more. The second is the fact that in the noise-free case uncertainty exists only for a positive prediction time. A negative prediction time, which corresponds, of course, to the determination of the value assumed by the signal at some time in the past, can be set into the analysis as easily as a positive prediction time, merely by shifting the impulsive admittance to the right rather than the left. In the noise-free case, however, there is nothing to be discarded when we shift to the right, since the impulsive admittance with which we begin is in any case identically zero for negative times. Thus the uncertainty in the determination of any past value of the signal is zero. Since we have postulated no noise to confuse the data, this is, of course, an inevitable result. As soon as noise is included, on the other hand, there is no such sharp distinction between the future and the past.[e] The uncertainty in the determination of the true value of the signal in the near past is almost as great as it is in estimating what the signal will be in the near future. As we go further

[e] This statement is to be understood in a physical rather than a mathematical sense. It is not intended to imply that there may not be sharp changes of behavior in the impulsive admittance at zero.

and further into the past the uncertainty gradually diminishes. If we can allow ourselves unlimited lag, we at length reach a point at which the discarded portion of the impulsive admittance characteristic is negligibly small. This, however, does not mean that all uncertainties have disappeared, but merely that we can base our estimate of the signal upon the power-ratio rule developed previously.

8.8 OVERALL CHARACTERISTICS OF PREDICTING NETWORKS

It has been fairly easy to develop a qualitative picture of the general characteristics of typical data-smoothing networks. As we have seen, they have amplitude characteristics of the low-pass filter type combined with lagging phase shifts. No corresponding qualitative picture of the characteristics of a typical overall predicting circuit has, however, been developed as yet. The discussion just concluded provides a rule for determining the characteristics of a predicting circuit in any given case, but provides comparatively little in the nature of a description of the result we may expect to secure.

In any particular situation we can, of course, calculate the overall characteristics of the predicting circuit. A simpler way of characterizing the overall predictor characteristic qualitatively, however, is based upon the use of the attenuation-phase relations for physical networks. We need merely use such an equation as (3) backward. Thus, we have previously shown that a positive phase slope corresponds to a lagging output. Correspondingly, a negative phase slope can be interpreted to represent a lead, or in other words, a prediction.[f]

[f] This, of course, does not mean that a network with a negative phase slope can predict a perfectly arbitrary event. We can hope to realize a negative phase slope, in combination with a flat amplitude characteristic, over only a finite band. The spectrum of an arbitrary event, that is, any suddenly applied signal, will always include important components running out to infinite frequency, where the negative phase slope can no longer be realized. The statement does, however, mean that if we suddenly apply a signal made up of one or more low-frequency sinusoids, *and wait for the steady state to become established,* the output will appear to lead the input by a time equal to the slope of the negative phase characteristic.

If we assign $(dB/d\omega)_{\omega=0}$ in equation (3) a negative value, we see that $A - A_0$ must on the average be negative. In other words, the amplitude characteristic of an overall prediction circuit must rise, on the average, as we proceed upward from zero frequency. This is in marked contrast to a data-smoothing network, which, as we have seen, tends to have a low-pass filter type of characteristic with a falling amplitude characteristic at high frequencies. The increased amplitude of response may have two detrimental effects. In the first place, it evidently produces a distorting effect on any signal components to which it applies. In the second place, it produces an exaggerated response to noise.

Examples of the characteristics of overall prediction circuits are readily constructed by reference to the circuit of Figure 21. Various

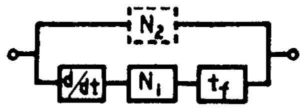


FIGURE 21. One-dimensional prediction circuit with data-smoothing networks.

particular results are obtained by assigning particular characteristics to the data-smoothing network. Thus, if the data-smoothing network is absent entirely the transmission through the path containing the differentiator is $i\omega t_f$, since differentiation is equivalent to multiplication by $i\omega$. The attenuation of the overall circuit is consequently $A = -\log |1 + i\omega t_f|$. This is plotted as curve I of Figure 22. The increasing amplitude characteristic at high frequencies is obviously due fundamentally to the increased transmission through the differentiator circuit.

If the data-smoothing network is assigned the characteristic $(1 + i\omega\alpha)^{-1}$, corresponding to a very simple low-pass filter type of response, the overall transmission becomes that shown by curve II in Figure 22. (It is assumed that $\alpha = t_f$, for simplicity.) The negative attenuation at high frequencies is much reduced. This is paid for by an increased amplitude of response at low frequencies, but since the integration in (3) takes place on an inverse frequency scale, the low-frequency fragment is much less than the gain reduction at high frequencies. Curve

III shows the result when the data-smoothing network is assigned the characteristic $(1 + i\omega\alpha)^{-2}$. Finally, curve IV shows the result obtainable when there is also a filter in the

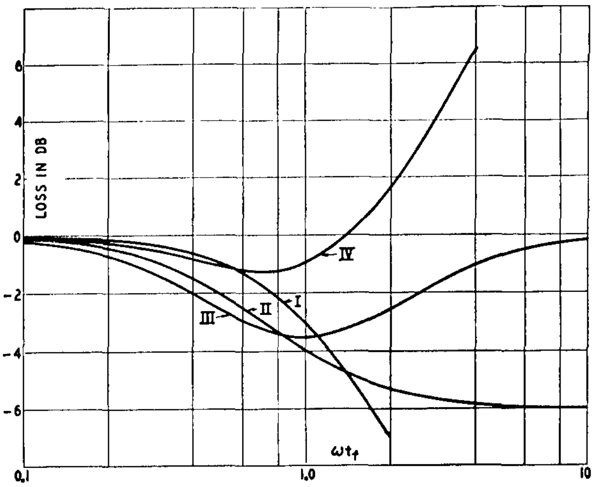


FIGURE 22. Attenuation characteristics of prediction circuit shown in Figure 21.

present-position circuit (as shown by the broken lines in Figure 21), so that there may be a net positive attenuation at high frequencies.

In view of the inverse frequency scale in (3), the gross negative attenuation will be minimized if the negative attenuation region is placed very close to zero frequency. This, however, means that much of the signal energy falls in the negative attenuation region so that in certain respects, at least, the signal response must be seriously injured. For example, in the specific circuits just discussed we can place the negative attenuation region at very low frequencies by choosing very long time constants, α, in the data-smoothing networks, with the consequence that the circuits will operate correctly for any long continued straight line path, but will be very sluggish in changing from one straight line to another. If the negative attenuation region is placed at higher frequencies, on the other hand, the signal response is improved but beyond certain limits the circuit becomes unbearably sensitive to noise.

Quantitative illustrations of these relationships are quickly constructed. Suppose, for example, that the prediction time is 2 seconds. From (3) this is consistent with an attenuation characteristic having zero attenuation below $\omega = 1$ and a net gain of π népers thereafter. In other words, the amplitudes of all frequencies below $\omega = 1$ are increased by a factor of about 22 to 1. If the region of added gain is pushed to a higher frequency or concentrated within a narrow band, the multiplying factor rapidly becomes larger. For example, if we maintain A at approximately zero below $\omega = 2$, the average gain above this point must be 2π népers, corresponding to a multiplying factor of 500 to 1. We secure the same factor by attempting to concentrate the region of negative attenuation in the band between $\omega = 1$ and $\omega = 2$. The multiplying factor also goes up rapidly as we increase the prediction time. For example, with the gain uniformly spread over the frequency region above $\omega = 1$ the multiplying factor is 500 for a prediction time of 4 seconds, or more than 10,000 for a prediction time of 6 seconds.

Reasonable multiplying factors with long prediction times can be obtained only by carrying the negative attenuation region to very low frequencies. As indicated previously, the cost of this is an increase in the time required for the signal to change from one constant or nearly constant value to another. For example, in the first illustration above, if the region of π népers net gain is carried down from $\omega = 1$ to $\omega = 0.2$ the integral in (3) is just five times as great as it was before, so that the characteristic corresponds to a prediction time of 10 rather than 2 seconds. This change would correspond to an increase[g] from perhaps 4 or 5 to perhaps 20 or 25 seconds in the time required for the circuit to settle from one constant value to another.

Practical examples of the transmission characteristics of overall prediction circuits, with particular emphasis on the dominant effect of even very small negative attenuations at extremely low frequencies, are shown later in Figures 5 to 8, inclusive. In the linear predictor, $A - A_0$ varies as $-k\omega^2$ nears zero, and it is easily seen that such a term makes a finite con-

[g] Only rough numbers can be given, since circuits with the square-cornered attenuation characteristics chosen for illustrative purposes would have very ripply transient characteristics, corresponding to no very well marked settling time.

tribution to the integral in (3). On the other hand, the attenuation of the quadratic predictor, which is capable of dealing exactly with polynomial functions of time of the second degree or less, is necessarily zero at the origin[h] to terms of the order of ω^4, so that the integral in this region can be neglected. This slight difference between the two characteristics at frequencies of the order of 0.01 cycle per second and below is sufficient to balance the obviously greater negative attenuation of the quadratic predictor at higher frequencies.

[h] Cf the discussion of Quasi-Distortionless Prediction Networks in Appendix A.

Chapter 9

THE ASSUMPTION OF ANALYTIC ARCS

THE DISCUSSION in the previous two chapters has been based upon the assumption that the least squares criterion forms a suitable measure of performance for a predicting network. This assumption permitted us to restrict our attention to the amplitude spectra of the signal and noise, leaving phase relations entirely out of account. Thus, both signal and noise could be thought of as "random noise" functions characterized by random phases and Gaussian distributions, as described in the preceding chapter. So far as the noise is concerned, there seems to be nothing wrong with this assumption. In the case of the signal, however, it appears that significant phase relations may exist. This chapter will consequently set up an alternative analysis which permits the significance of possible phase relations in the target paths to be estimated.

The alternative analysis is based upon the assumption that the target courses are sequences of analytic segments of different lengths joined together. These segments are simple predictable curves such as straight lines, parabolas, and circles. Significant phase relations are implied by the assumption that there are sudden changes from one type of course to another.

This picture of target paths is, of course, extreme. There are no such sharp discontinuities between one segment and another, nor do airplanes fly perfectly along simple curves even for limited periods. Nevertheless, it is the conception of target courses upon which the rest of our analysis is based. The reasons for believing that it is a closer approximation to actual target courses than, say, a random noise function with the same power spectrum would be, are given later. Perhaps more important is the fact that the possibility of hitting an airplane flying along such a simple analytic arc is much greater than it would be if we were attempting to predict a corresponding random noise function. It is thus advantageous to take the analytic arc assumption as a basis for designing the prediction circuit, even if the assumption seems to be reasonably well justified over only occasional segments of actual target paths. An example of such a situation is furnished by the bombing run illustration described in Chapter 7.

As a corallary to the analytic arc assumption it is also assumed that the theoretical predicted point must be quite close to the actual target position if the probability of scoring a hit is to be appreciable. In other words, such dispersive factors as random errors in computer or gun or the lethal radius of the shell, which would tend to produce occasional hits at long distances from the theoretical predicted point, are quite small. This is such a plausible assumption in the light of present-day antiaircraft experience that its critical importance in the present argument is likely to go unperceived. However, this is the assumption which limits consideration to small errors in prediction, whereas the least squares criterion naturally gives greatest emphasis to large errors. If, for example, antiaircraft projectiles were suddenly endowed with a much greater destructive radius, we would be much more interested in fairly large misses, and the objections to the least squares criterion would disappear.

These postulates are discussed in more detail in the following sections. In anticipation of this discussion the following conclusions may be mentioned:

1. With the assumptions as stated, the prediction should be on a modal rather than a least squares basis. In other words, the gun should be aimed at the most probable future position of the target.

2. Modal prediction requires evaluation of the parameters of the analytic arc the target is at present traversing. This can be accomplished by smoothing the values of these parameters evaluated for a period in the past.

3. If the smoothing is performed by linear invariable networks, the impulsive admittances of these networks should have a definite cutoff after a finite smoothing time. By this means

all data over a certain age are given zero weight. The method of calculating the proper smoothing time is developed.

4. Definite advantages can be obtained from circuits with variable smoothing times if such systems can be satisfactorily mechanized.

9.1 THE TARGET COURSES

The target courses, like the tracking errors, can be thought of as a statistically generated set of functions—that is, a stochastic process. The structure of this process is, however, very different from that of the tracking errors. It is by no means satisfactory to assume the target courses to be equivalent to a random noise having the same power spectrum as the target courses. As we pointed out in Chapter 7, the target is piloted by a purposeful human being. It tends to follow a definite simple curve for a period of time and then to shift to a new simple curve. Much of the flight is in attempted straight lines with constant velocity. Most of the remainder can be considered to be segments of circles or helices in space, or as segments of parabolas or higher degree curves. Straight line constant speed flight corresponds to the airplane controls in a neutral position. The helical flight is a natural generalization allowing arbitrary, but fixed, positions of the controls. The curves which are parabolic functions of time correspond to constant acceleration in the three space coordinates. Thus, all these assumptions have a reasonable physical background.

Most antiaircraft computers are constructed on the assumption of straight line flight, although some work has been done in World War II on curved flight directors both with the helical and the parabolic assumptions. There is not a great deal of difference in these two generalizations from the practical point of view, since determination of acceleration terms is subject to such large errors in any case.

The important part of this representation of the target courses is that they consist of segments of simple analytic curves joined together. The individual segments are completely predictable if we have a part of the segment given exactly. One need merely evaluate the parameters of the segment from the given part and evaluate the curve for $t = t_f$. The unpredictable part of the target courses is due to the possibility of sudden changes from one segment to another. With random noise functions the unpredictableness occurs continuously.

This simplified description of the target courses as piecewise analytic functions must be recognized as only a first approximation. A more complete description of the target course would include the "fine structure," the connecting curves between the various analytic segments and the deviations from the segments due to random air disturbances and similar causes. This latter effect, the wandering of the target from its intended path, might be reasonably well represented by the addition of a random noise function to the piecewise analytic functions described above.

9.2 THE POISSON DISTRIBUTION OF SEGMENT END POINTS

The analytic segments of which the course is supposed to consist are not all of the same duration — we may assume some probability distribution of the duration of these segments. The simplest assumption here is that the breaks occur in a Poisson distribution in time. This assumption is not necessary for our analysis but is a reasonable one and leads to a simple mathematical treatment. Any other reasonable distribution would give comparable results.

A series of events is said to occur in a Poisson distribution in time if the periods between successive events are independent in the probability sense and are controlled by a distribution function

$$p(l)dl = \frac{1}{a} e^{-l/a} dl .$$

Here $p(l)\,dl$ is the probability of an interval of length between l and $l + dl$. This means that the frequency of intervals of a given length is a decreasing exponential function of the length. This type of distribution is familiar in physics as describing the decay of radioactive substances. The time a in the distribution function is the average length of the intervals, since

$$\bar{l} = \int_0^\infty l\, p(l)dl$$

$$= \int_0^\infty \frac{l}{a} e^{-l/a} \, dl$$

$$= a \, .$$

It is related to the "half life" b of the interval by

$$b = a \ln 2 \, .$$

The single number a completely specifies the Poisson distribution. The events may be said to be happening as randomly as possible apart from the fact that they occur at an average rate of $1/a$ per second.

Another way of describing a Poisson distribution of events is the following. The probability of an event in a small interval of duration dl is $(1/a)\,dl$ and is independent of whether or not events have occurred in any other nonoverlapping intervals.

9.3 THE PROBABILITY DISTRIBUTION OF FUTURE POSITIONS

Let us suppose that we have a record of the course of the target up to the present time and a complete statistical description of the set of target courses. What can then be said about the position of the target t_f seconds from now? If we were able to analyze the data completely the most we could obtain would be a probability distribution function for the future position. This distribution function would give the probability, in the light of the course history, of the target being at any point in space at the future time. This function would assume large values at likely points and low values at unlikely points. For t_f small the distribution would be highly concentrated and for larger t_f it would tend to spread out.

In the simple case we have been discussing, of a Poisson distribution of sudden changes in type of course, the distribution consists of two parts. First, there is a spike of probability at one point, the continuation of the present predictable segment. Second, there is a continuous distribution which corresponds to possible changes to a new segment during the time of flight. As t_f increases the total probability in the spike decreases exponentially toward zero, and the total in the continuous part increases exponentially toward unity. The behavior is roughly as indicated in Figure 1.

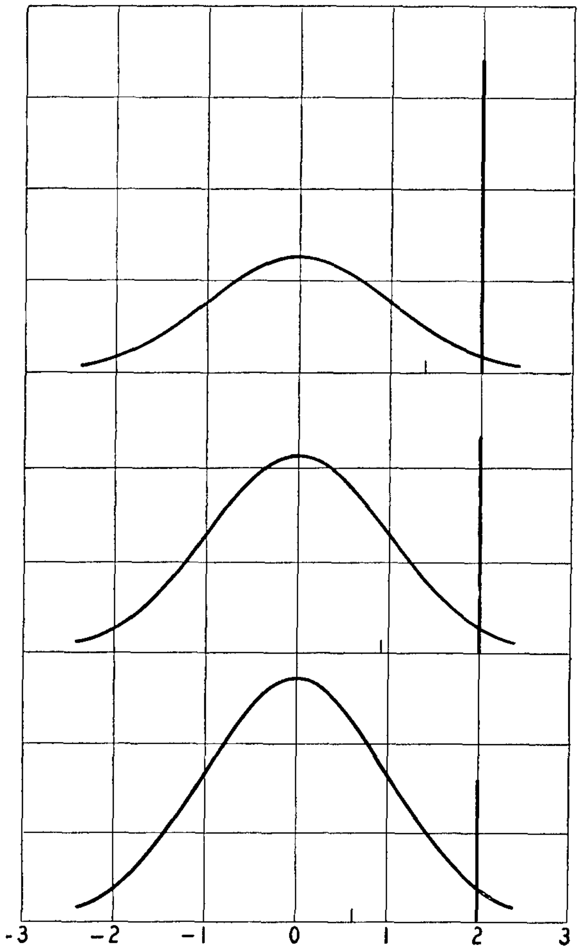


FIGURE 1. Probability distribution of future position of target, assuming piecewise analytic courses.

A very different type of future position distribution is exhibited with other assumptions about the target courses. For example, suppose the courses were random noise functions with the power spectrum

$$P(\omega) = \frac{1}{a^2 + \omega^2} \, .$$

A typical noise function with this spectrum is shown in Figure 2. In Figure 3 is shown a typical velocity under the other assumption, that the courses are piecewise analytic and in fact straight lines between breaks. If the breaks are Poisson distributed, both Figure 2 and Figure 3 have the same power spectrum, $1/(a^2 + \omega^2)$. The future distribution of velocities for Figure 3 is shown in Figure 1, and for Figure 2, it will be as shown in Figure 4. In the random noise case the future distribution is a

Gaussian distribution with no spike. The center of this distribution decreases exponentially toward zero with increasing time of flight according to the formula

$$\overline{X}_{t_f} = X_0 e^{-at_f}$$

where X_0 is the present value of the function and $\overline{X}_{t_f}$ is the mean of the future distribution.

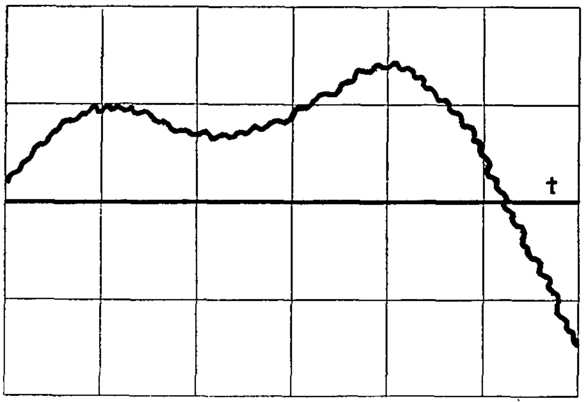


FIGURE 2. Typical noise function.

The standard deviation σ of the distribution increases exponentially toward the rms value of the function according to

$$\sigma = A(1 - e^{-2at_f}).$$

Supposing that this distribution function could be determined, where should the gun be aimed? The answer to this will depend on two factors: the gun dispersion, and the lethal

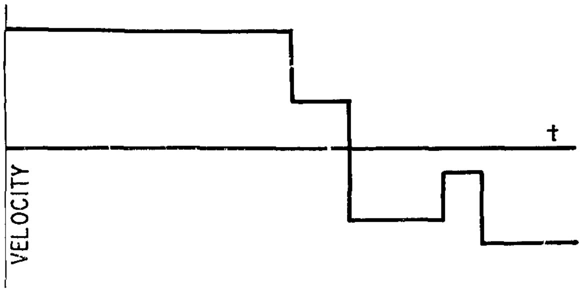


FIGURE 3. Typical velocity function.

effects of the shell. If the gun is aimed to explode the shell at a certain point in space, the shell will not necessarily explode at that point, but rather there will be a distribution of positions centered about the point aimed at, because of gun dispersion. Also, if the shell explodes at a certain point and the target is at another point, there will be a certain probability of lethal effect which decreases rapidly with increasing distance between the points. These two functions could be combined by a product integration to give the probability of lethal effect if the target is at one point and

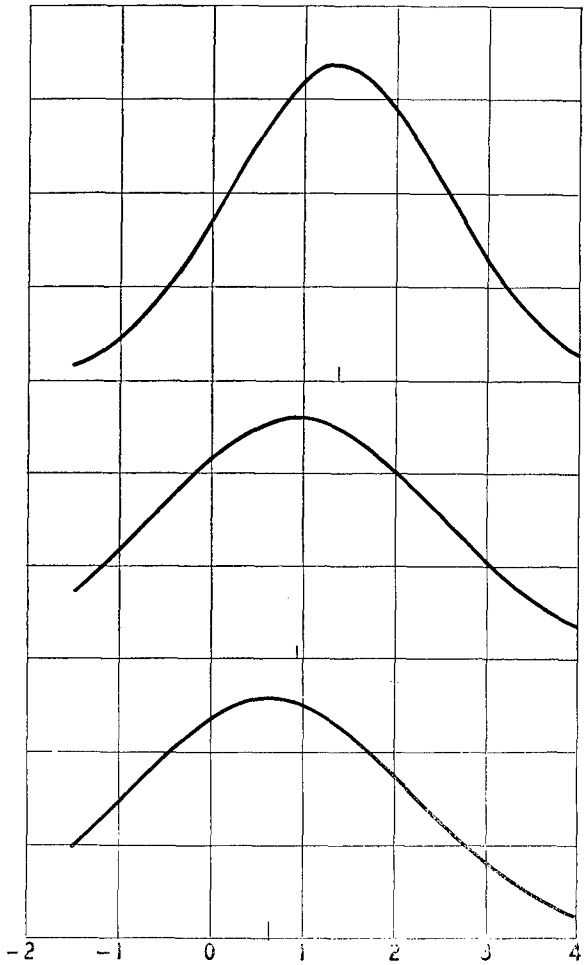


FIGURE 4. Probability distribution of future position of target, assuming courses with random noise properties.

the gun aimed to explode the shell at a second point. To determine the probability of a hit when aiming at a certain point, then, we should multiply the probability of the target being at each point in space by the probability of lethal effect when it is at that point and integrate the product over all space. The optimum point of aim will be the one which maximizes this integrated product.

In one dimension this may be expressed mathematically as follows. Let $P(x)$ be the future position distribution of the target, so that $P(x)dx$ is the probability of it being in the interval from x to $x + dx$ at the future time. Let $Q(x,y)$ be the probability of hitting the target if the gun is aimed at point y and the target is at point x. Then the total probability of a hit when aiming at point y is

$$R(y) = \int P(x)\ Q(x,y)\ dx\ .$$

The point of aim y should be chosen to maximize $R(y)$.

In the cases we consider, the lethal radius of the shell and the dispersion of the gun are both assumed to be small in comparison with the range of future positions if there is a change of course during the time of flight. This means that $Q(x,y)$ is small unless x is xery near to y. $Q(x,y)$ can be, in fact, considered to be a δ function of $(x\text{-}y)$, and the value $R(y)$ is then just a constant times $P(y)$. Thus, *the best aiming point under this assumption is the most probable future position of the target.* The assumption of small lethal distance is generally valid with antiaircraft fire and ordinary chemical explosive shells.

Now the most probable future position in our case is the spike of probability corresponding to the analytic extrapolation of the present segment of the target course. To determine its position one must find the parameters of this segment and evaluate for t_f seconds in the future. For example, if the segments are assumed to be straight lines (constant velocity target) the velocity components are determined and multiplied by t_f to give the predicted change in position. These changes are added to the present position to give the future position. If helical or parabolic segments are assumed, the parameters of these curves are determined from the past data, and the curves extrapolated t_f seconds into the future.

These conclusions may be contrasted with the idea of aiming at the point which minimizes the mean square error. The least squares criterion amounts to aiming at the mean or center of gravity of the future distribution of position. This point will ordinarily be under the continuous part of the distribution and not at the spike; e.g., the point marked in Figure 1. Its position depends to a considerable extent on distant parts of the distribution, which would surely be complete misses in any case. The chief advantage of the least squares criterion is that it fits in well with the mathematical tools suitable to these problems, leading to solvable equations.

The least squares criterion will still appear in our analysis in that we attempt to smooth our *course parameters* in such a way as to minimize the mean square error in *these,* a very different thing from minimizing the mean square error in the predicted position of the target.

9.4 NECESSITY OF A SHARP CUTOFF

The changes in the course parameters between adjacent segments can be very large. Also, at the start of operations and in changing from one target to another there will be large and erratic variation of the input to the smoothing and predicting circuits, unrelated to the present target course. If any of these data are used in prediction, the result will almost surely be a miss because of the small lethal radius of the shell. The only way to eliminate these errors in a linear invariable system is to have all weighting functions cut off sharply after a short time. Then all data over a certain age are eliminated. Hits will occur only when the target has been on a predictable segment for this length of time or more and remains there at least t_f seconds in the future.

Suppose the weighting function for velocity has a 1 per cent tail beyond the cutoff point and that the trackers start following the target from a zero position. Then after the smoothing time there will be, because of the lack of exact cutoff, a 1 per cent error in velocity. If the time of flight were 15 seconds and the target velocity 200 yards per second, this represents an error of 30 yards in predicted position. Since this is comparable to the other errors in a typical director, we conclude that the tail of the smoothing curve should not be much greater than 1 per cent of its total area.

9.5 CALCULATION OF THE BEST SMOOTHING TIME

Under the assumptions we have made, the proper smoothing time to maximize the number of hits can be determined as follows. Let $P(l)$

be the probability that a predictable segment of the course lasts for l seconds or more. In the Poisson case this function is

$$P(l) = e^{-l/a} .$$

With a given smoothing time S there will be a certain probability of hitting the target, assuming it has been on the present segment for S seconds in the past and will remain there for t_f seconds in the future. We assume changes in course to be so large that any change results in a miss. This probability of a hit $Q(S)$, provided it remains on the course, will be an increasing function of S. Ordinarily the standard deviation will decrease as the square root of the smoothing time. We have assumed the lethal radius of the shell small compared to the dispersion of shells about the target. The probability of a hit will then vary inversely with the volume through which the shells are dispersed. If the gun itself had no dispersion but all errors were due to tracking errors (and if the tracking error spectrum is flat), the probability of a hit would then vary as $KS^{3/2}$ for S in the region of interest. This is because there are three dimensions and the expected error in each of these is decreasing as $S^{-1/2}$. With gun dispersion present, $Q(S)$ will have the form

$$Q(S) = K\left(\sigma_1^2 + \sigma_2^2 \frac{a}{S}\right)^{-3/2}$$

where σ_1 is the standard deviation due to the gun dispersion, and $\sigma_2\sqrt{a/S}$ that due to tracking errors. The sum of the squares is the total variance in each dimension and the three-halves power gives the total dispersion volume.

When these two functions $P(l)$ and $Q(S)$ are known, the best smoothing time is that which minimizes the product

$$P(S + t_f) \cdot Q(S) .$$

The first term is the probability of a predictable segment of the course lasting $S + t_f$ seconds, and the second term is the probability of a hit if it does last that long. Therefore, the product is the probability of a hit with smoothing time S.

In the Poisson case, with no gun dispersion, the calculation is as follows:

$$P(l) = e^{-l/a}$$

$$P(S + t_f) = e^{-\frac{S+t_f}{a}} = Ae^{-S/a}$$

$$Q(S) = \sigma S^{3/2}$$

$$f(S) = P(S + t_f)Q(S) = Be^{-S/a} S^{3/2}$$

$$f'(S) = B\left[e^{-S/a}\frac{3}{2}S^{1/2} - \frac{1}{a}e^{-S/a}S^{3/2}\right] = 0$$

$$S = \frac{3}{2}a$$

The proper smoothing time is 3/2 of the average segment length, and is independent of the time of flight and all other factors.

The presence of gun dispersion and computer errors which are independent of smoothing time decreases the best S from this value. In this case the equation for optimal S is the quadratic

$$\sigma_1^2\left(\frac{S}{a}\right)^2 + \sigma_2^2\frac{S}{a} - \frac{3}{2}\sigma_2^2 = 0 ;$$

hence

$$\frac{S}{a} = \frac{-\sigma_2^2 + \sigma_2\sqrt{\sigma_2^2 + 6\sigma_1^2}}{2\sigma_1^2}$$

$$= \frac{\sigma_2}{2\sigma_1}\sqrt{\left(\frac{\sigma_2}{\sigma_1}\right)^2 + 6} - \frac{1}{2}\left(\frac{\sigma_2}{\sigma_1}\right)^2.$$

Here σ_1 is the part of the errors which is independent of smoothing time (dispersion errors in the computer, etc.) and σ_2 is the error which varies inversely with the square root of S, σ_1 being its value at $S = a$. Ordinarily σ_1 is several times σ_2 in which case we have approximately

$$\frac{S}{a} = \frac{\sigma_2}{\sigma_1}\sqrt{\frac{3}{2}} .$$

There are other factors which we have neglected, which decrease the best smoothing time still further. The wandering of the target about the predictable segments assumed in the above simplified analysis makes old data less reliable and therefore reduces S. Also, there is the tactical consideration that when starting to track a target it is desirable to commence firing as soon as possible, even if reducing this time makes individual hits somewhat less probable. For these and other reasons the best smoothing time will be just a fraction of a.

9.6 NONLINEAR AND VARIABLE SYSTEMS

The compromise required in choosing a certain definite smoothing time can be eliminated by the use of nonlinear elements. In particular, if a method is devised for determining when changes of course occur, this indication can be used to start a new linear but variable smoothing operation, so that the device uses all the data pertinent to the present segment and no data from previous segments. There is a clear improvement in such cases although not so great as might be expected. There are many practical difficulties in proper adjustment of such a "trigger" action. If the trigger is too sensitive it will assume new segments due merely to tracking noise and seldom allow sufficient smoothing for accurate fire. If it is too insensitive it fails in its function of quickly locating changes of segment. Since the noise and target courses are subject to considerable variation, this adjustment is not easy.

In such a system the smoothing may be linear—the only nonlinearity is the tripping circuit. The analysis of best weighting functions, etc., given in later chapters can for the most part be applied to such cases. There may also be advantages to be derived from making the smoothing operator depend on the general position in space of the target relative to the gun. The smoothing time may be varied, for example, as a function of the time of flight. This type of variation would be slow compared to the noise frequency, and here again the linear analysis can be used.

Whether any real advantage can be obtained by "strongly" nonlinear smoothing in practical cases other than these two possibilities is questionable.

Chapter 10

SMOOTHING FUNCTIONS FOR CONSTANTS

THE ANALYTIC ARC ASSUMPTION described in the previous chapter immediately allows us to reduce a vast proportion of data-smoothing problems to a relatively concrete form. Obviously the arc will be specified by a number of parameters and the principal object of the computing and data-smoothing circuits must be to isolate values of these parameters on the basis of which a prediction can be made. In practical cases the instantaneous values of the parameters are isolated by coordinate converters. The function of the data-smoothing circuit is to provide a suitable average from these instantaneous values. This is called "smoothing a constant" here since the parameters are assumed to be constant along each arc, although they may change radically from one arc to another.

The data-smoothing network is most conveniently specified by its impulsive admittance. (See Appendix A.) In accordance with the assumptions made in the previous chapter, it will be assumed that the desired impulsive admittance is identically zero after some limiting time T. Thus, T seconds after a change from one analytic arc to the next the new parameter value is established. T is the so-called "settling time" of the data-smoothing network.

With the settling time limit given, the problem of choosing a suitable data-smoothing network reduces to that of finding the best shape of the impulsive admittance characteristic for $t < T$. Obviously this shape determines how the output of the network changes in going from the parameter value appropriate for the first arc to that appropriate for the second. The exact way in which the response settles from one constant value to the next is, however, usually of comparatively little interest. The shape of the weighting function is of importance chiefly because of its effect on the noise. For each noise spectrum there is, in principle, an optimum shape for the weighting function. The present chapter approaches the problem of choosing a shape which will minimize the effect of noise from several points of view.

It should be noted that the term noise as used here does not necessarily refer to the errors associated directly with the tracking data. The tracking data may have been subjected to coordinate conversions, differentiations, or other processes of computation before reaching the data-smoothing network.[a] The noise associated with the signal to be smoothed thus will usually have characteristics differing from those of the noise associated with the tracking data.

10.1 EXPONENTIAL SMOOTHING

Before attacking the problem of smoothing a constant in a systematic way it is worth while to consider an important special case. This is the so-called exponential smoothing circuit. It leads to a data-smoothing network in which the output V is related to the input E by

$$V(t) = \alpha \int_0^\infty E(t - \tau)\, e^{-\alpha\tau}\, d\tau$$

so that the impulsive admittance $W(t)$ is an exponential function of time, as illustrated by Figure 1.

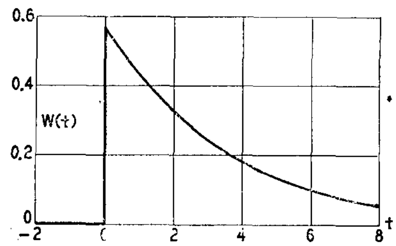


FIGURE 1. Simple exponential weighting function.

An impulsive admittance of the type shown in Figure 1 does not show any very definite settling time. The exponential curve approaches zero gradually, and it is a long time after a change in course before the effects of the data obtained on the old course are negligible. This is obviously an undesirable result,

[a] In exceptional circumstances the physical apparatus in which these processes are carried out may also be sources of additional noise.

and the exponential weighting function is consequently not a recommended one for situations to which the analytic arc assumption applies. The exponential solution is, however, described here because it occurs in such a vast variety of cases. It is found, in fact, whenever the data-smoothing device is specified by a linear first-order differential equation with constant coefficients. It may thus correspond to many simple situations. For example, this is the result which would be obtained in an electrical circuit if we smoothed the data by placing a simple shunt capacity across a resistance circuit. In mechanical structures it is encountered whenever the damping depends either upon simple inertia or a simple compliance.

Simple exponential smoothing also occurs in a variety of other situations which may be somewhat less obvious. For example, it is the effective result in either an aided laying or a regenerative tracking scheme whenever the ratio between rate and displacement corrections is fixed. Another somewhat similar example is furnished by the feedback amplifier circuit shown in Figure 2. Since rapid fluctuations in the output of this amplifier are fed back through the capacity and tend to oppose the input voltage, the structure acts as a smoother, and more detailed analysis would show that it has characteristics similar to those obtained by using a shunt capacity across a resistance circuit. The structure is introduced here because considerable use is made of it in connection with the discussion of nonlinear smoothing in a later chapter.

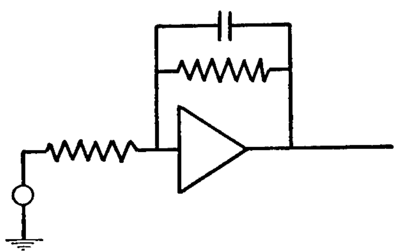


FIGURE 2. Feedback amplifier circuit giving simple exponential weighting function.

One simple conclusion about data-smoothing networks can be drawn immediately from this discussion. Since all structures simple enough to be specified by a first-order differential equation give exponential smoothing, which has no very well-marked settling time, it is clear that a data-smoothing network which shows a well-defined settling time must probably be at least moderately complicated.

10.2 CURVE-FITTING METHOD

Consider the signal E shown in Figure 3 under the assumption that the true signal is constant and the superposed noise is random

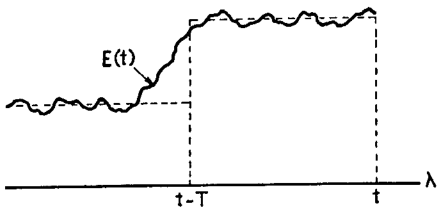


FIGURE 3. Piecewise constant signal with noise.

with a flat spectrum. The best constant A, in the least squares sense, which can be fitted to the signal from $t - T$ to t is that which minimizes

$$\int_{t-T}^{t} [A - E(\lambda)]^2 \, d\lambda \,,$$

viz.,

$$A = \frac{1}{T}\int_{t-T}^{t} E(\lambda)\, d\lambda \,. \tag{1}$$

Comparing this with equation (2), Appendix A, it will be seen that A, which is obviously a function of t, is the response to the assumed signal of a network whose impulsive admittance is

$$W(t) = \frac{1}{T} \quad 0 < t < T \,. \tag{2}$$

This is the best weighting function for smoothing under the assumed circumstances. It is illustrated in Figure 4.

A more complex situation is one in which the true signal is a line of constant slope with

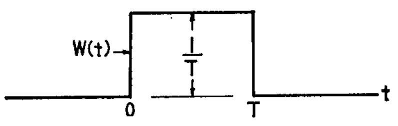


FIGURE 4. Best weighting function for smoothing piecewise constant signal.

superposed flat random noise, as shown in Figure 5. For convenience the analysis will be conducted in terms of the age variable $\tau = t - \lambda$.

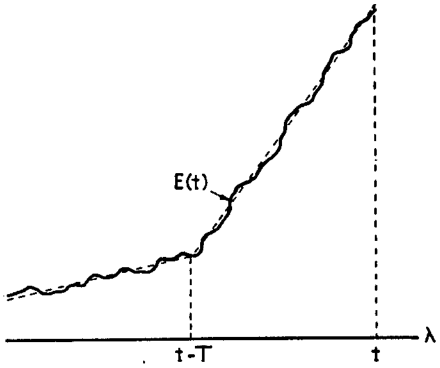


FIGURE 5. Piecewise linearly varying signal with noise.

The best straight line $A - B\tau$ which can be fitted to the signal from $\tau = 0$ to $\tau = T$ is that which minimizes

$$\int_0^T [A - B\tau - E(t-\tau)]^2 \, d\tau .$$

Hence A and B must satisfy simultaneously

$$A - \frac{T}{2} B = \frac{1}{T}\int_0^T E(t-\tau)\, d\tau$$

$$\frac{T}{2} A - \frac{T^2}{3} B = \frac{1}{T}\int_0^T E(t-\tau)\,\tau\, d\tau . \qquad (3)$$

Eliminating A, we get

$$B = \frac{12}{T^3}\int_0^T E(t-\tau)\cdot\left(\frac{T}{2}-\tau\right) d\tau,$$

whence by partial integration

$$B = \frac{6}{T^3}\int_0^T E'(t-\tau)\cdot\tau(T-\tau)\cdot d\tau .$$

Comparing this with (7), Appendix A, it will be seen that B, which is obviously a function of t, is the response to the derivative of the assumed signal of a network whose impulsive admittance is

$$W(t) = \frac{6}{T}\cdot\frac{t}{T}\left(1 - \frac{t}{T}\right) \quad 0 < t < T . \qquad (4)$$

This is the best weighting function for smoothing the derivative of the signal under the assumed circumstances. It is illustrated in Figure 6 and is generally referred to as the "parabolic weighting function."

It should be noted also that the right-hand member of the first of equations (3) is formally the same as that of equation (1). Hence the response of the network specified by (2)

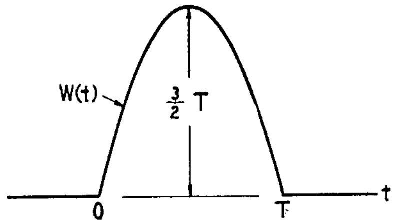


FIGURE 6. Best weighting function for smoothing piecewise linearly varying signal.

and illustrated in Figure 4, to the type of signal shown in Figure 5, will correspond to the value on the best straight line $T/2$ seconds back from t, the present time. This network is still the best for smoothing the signal, but it introduces a delay of one half of the smoothing time. The delay may be reduced only at the price of a reduction in smoothing unless the smoothing time is increased.

10.3 AUTOCORRELATION METHOD

The autocorrelation method with finite settling time was first used by G. R. Stibitz in numerical determination of the best weighting function for smoothing the derivative of tracking data with typical tracking errors. This method was also used to determine the sensitivity of smoothing to departures of the weighting function from the best form.

The analysis is based upon the formula

$$V(t) = \int_0^T g'(t-\tau)\cdot W(\tau)\, d\tau \quad t > T$$

for the response to the derivative of the error time function $g(t)$ of a network whose impulsive admittance or weighting function $W(t)$ is identically zero for $t > T$ as well as for $t < 0$. Since measured tracking errors are generally tabulated only at 1-second intervals, the integral may be approximated by the sum

$$V(t) = \sum_{m=1}^{T} \Delta g_{t-m+(1/2)} \cdot W_{m-(1/2)}$$

for integral values of t.

The instantaneous transmitted power is the square of this expression, and the average transmitted power is

$$P_{avg} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N} V^2(t) .$$

This may be expressed in the form

$$P_{avg} = \sum\sum_{m,n=1}^{T} W_{m-(\frac{1}{2})} \cdot C_{m-n} \cdot W_{n-(\frac{1}{2})} \tag{5}$$

where

$$C_{m-n} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N} \Delta g_{t-m+(\frac{1}{2})} \cdot \Delta g_{t-n+(\frac{1}{2})}$$

is the autocorrelation of the errors. Having computed the autocorrelation, (5) may be minimized with respect to the W's by familiar methods, under the constraint

$$\sum_{m=1}^{T} W_{m-(\frac{1}{2})} = 1 .$$

The values of W thus obtained are the specification of the best weighting function.[b] Equation (5) may then be used to determine the sensitivity of smoothing to departures of the weighting function from the best form.

Proceeding along this line, Stibitz found that the best weighting function for typical actual tracking errors was generally intermediate to the uniform and parabolic ones shown in Figures 4 and 6. Furthermore, Stibitz found that the difference in smoothing obtained from the best weighting function on the one hand and from the uniform or the parabolic weighting function on the other hand, is negligible in practice.

The autocorrelation method was later formalized by R. S. Phillips and P. R. Weiss who incorporated it into a theory of prediction.[7] A brief exposition of this formulation is given in Appendix B.

10.4 ELEMENTARY PULSE METHOD

For the purposes of this method, an elementary noise pulse is defined by a time function $F_0(t)$ which satisfies the following requirements:

1. Identically zero when $t < 0$.
2. Contains no terms which increase exponentially with time.
3. Power spectrum $N(\omega^2)$ is the same as that of the noise.

The noise is then regarded as the result of elementary noise pulses started at random. Alternatively, it may be regarded as the result of flat random noise passed through a network whose transmission function is $S(p) = L[F_0(t)]$. As a matter of fact, only $S(p)$ is required in the analysis, and this is readily determined from the relation

$$|S(i\omega)|^2 = N(\omega^2) ,$$

together with the condition that $S(i\omega)$ corresponds to the transmission function of a minimum-phase physical structure (cf. Appendix B).

The response $F(t)$ to the elementary noise pulse $F_0(t)$ of a network whose impulsive admittance is $W(t)$ is given by the operational equation

$$F(t) = S(p) \cdot W(t)$$

in accordance with the footnote in Section A.5, Appendix A. The best form for $W(t)$ is therefore that which minimizes the integral

$$\int_{0-}^{\infty} [F(t)]^2 \, dt \tag{6}$$

under the restriction

$$\int_{0-}^{t_0} W(t) \, dt = 1 \tag{7}$$

when $t_0 > T$.

This is as much of the elementary pulse method as we shall need in order to reconsider the cases treated in Section 10.2. For the treatment of more general cases the method is described in greater detail in Appendix B.

The minimization of the integral (6) under the restriction (7) reduces to a simple isoperimetric problem in the calculus of variations, in cases in which $S(p)$ is a polynomial in p. It is essential first of all, however, to note that if $S(p)$ is of degree n, the integral (6) will converge only if $W(t)$ is differentiable at least n times. In other words, $W(t)$ must have continuous derivatives of all orders up to the $(n-1)$th inclusive, although the nth derivative may have finite discontinuities. In particular, if $W(t)$ is to be zero outside of $0 \leq t \leq T$, its

[b] The computations involved may be considerably reduced by noting the symmetry property proved in Section B.2, Appendix B.

derivatives of orders up to the $(n-1)$th inclusive must vanish at both $t = 0$ and $t = T$. These $2n$ boundary conditions must be imposed on the solution of the Euler equation which in this case is

$$S\left(\frac{d}{dt}\right) \cdot S\left(-\frac{d}{dt}\right) \cdot W(t) = \lambda \, .$$

λ is a constant parameter which is finally adjusted to that the restriction (7) is satisfied.

The first case treated in Section 10.2 is one in which $N(\omega^2) = 1$, whence $S(p) = 1$ and $F(t) = W(t)$. The integral (6) is a minimum under the restriction (7) if $W(t)$ is constant by intervals. The restriction (7) then requires $W(t)$ to be of the form (2).

The case of first derivative smoothing treated in 10.2 is one in which $N(\omega^2) = \omega^2$, whence $S(p) = p$ and $F(t) = \dot{W}(t)$. If the integral (6) is to converge at all, $\dot{W}(t)$ must not have discontinuities of impulsive or higher type; in other words, $W(t)$ must be continuous through all values of t. The integral is a minimum under the restriction (7) if $\ddot{W}(t)$ is constant by intervals. The restriction (7) then requires $W(t)$ to be of the form (4).

These results may be generalized immediately. In whatever way the signal to be smoothed may have been derived from the tracking data, let the power spectrum of the noise associated with it be $N(\omega^2) = \omega^{2n}$. Then $S(p) = p^n$ and $F(t) = W^{(n)}(t)$. If the integral (6) is to converge at all, $w^{(n-1)}(t)$ must be continuous through all values of t. The integral is a minimum under the restriction (7) if $W^{(2n)}(t)$ is constant by intervals. The restriction (7) then requires $W(t)$ to be of the form

$$W(t) = \frac{(2n+1)!}{(n!)^2 T} \left[\frac{t}{T}\left(1 - \frac{t}{T}\right)\right]^n \quad 0 < t < T \, . \tag{8}$$

It may be noted that the convergence requirements which arise in the foregoing discussion are directly related to the discussion and theorem in Section A.8, Appendix A, with respect to the relationship between discontinuities in the impulsive admittance and its derivatives on the one hand, and the ultimate cutoff characteristic of the transmission function on the other hand. The continuity of $W^{(n-1)}(t)$ is obviously required to make the transmission fall off ultimately at the rate of $6(n+1)$ db per octave against the rise of $6n$ db per octave in the noise power spectrum.

The integral (6) may also be used to evaluate the relative advantage of the best weighting function over another weighting function. As an example, consider the case where the weighting function (2) is the best. The value of the integral (6) in this case is $1/T$. If the weighting function (4) is used against the same noise, the value of the integral (6) is $6/5T$. Hence, as far as rms error or standard deviation is concerned, the second weighting function is $\sqrt{5/6}$ or 0.913 as efficient as the first.

Chapter 11

SMOOTHING FUNCTIONS FOR GENERAL POLYNOMIAL EXPANSIONS

THE THEORY of "smoothing a constant" developed in the preceding chapter will be extended in this chapter to the problem of smoothing a polynomial function of time of any prescribed degree. The extension is, however, restricted to the case of a flat noise spectrum. In addition to the smoothing problem, the analysis also provides a way of designing a network which will extrapolate the polynomial a given distance t_f into the future. The network is so arranged that t_f is continuously variable. In addition, the degree of the polynomial can readily be changed to fit changes in the complexity of the assumed form of the data, apart from noise.

It is clear that these results amount, in a certain sense, to an alternative to Wiener's method for the design of prediction circuits for general time series. Thus, to predict a time series of any given complexity we would need only to begin with a polynomial of sufficiently high degree to fit the observed data, and extrapolate. Aside from the restriction to a flat noise spectrum, perhaps the most obvious difference from Wiener's method is the fact that the settling time restriction limits the data upon which the prediction rests to a finite interval in the past. To advance such a prediction theory seriously, however, it would be necessary to go much farther into the way in which the degree of the polynomial is established and the justification for assuming that the extrapolated value represents a probable future value for the function.[a]

This general discussion will not be undertaken here. Since prediction with high degree polynomials will certainly be sensitive to minor irregularities in the data, tracking errors would necessarily limit the application of the method in any case. If we confine ourselves to reasonably low degree polynomials, however, the method is useful. An example is furnished by the prediction of airplane position, in rectangular coordinates, by quadratic functions of time. Here the square terms represent the effects of accelerations in the various coordinates. We can defend the inclusion of such terms on the ground that it is plausible to assume that an airplane may experience constant accelerations, due to turns, the force of gravity, etc., for considerable periods of time. The linear term represents plane velocity and needs no defense. The constant term, of course, gives the plane position at some reference time. Including it in the smoothing operation is equivalent to introducing "present-position" smoothing of the sort suggested by the broken lines in Figure 1 of Chapter 7.[b]

Aside from its direct interest as a possible prediction method, the analysis in this chapter is also of indirect interest for the additional light it sheds on the effect of the noise spectrum on smoothing functions. It turns out that smoothing a power of time, with a flat noise spectrum, is equivalent to smoothing a constant with a somewhat different noise spectrum. Thus the smoothing functions developed for polynomials are also useful as special cases of smoothing functions applicable to constants.

11.1 GENERAL METHOD

Let λ be any past value of time and let t be the present value. If the data is fitted with a smooth curve $\bar{E}(\lambda)$, the predicted value may be taken as $\bar{E}(t + t_f)$. The procedure of fitting is the familiar one of minimizing the integral

$$\int_{-\infty}^{t} [\,\bar{E}(\lambda) - E(\lambda)\,]^2\, W_0(t,\lambda)\, d\lambda$$

[a] As an example of possible difficulties we may notice the fact that two polynomials of different degree which approximate a given function as closely as possible, in a least squares sense, in a prescribed interval frequently differ radically outside that interval.

[b] In the circuit of Figure 1, Chapter 7, however, the smoothing network would produce a lag in the present-position data delivered to the prediction circuit, and this lag would, of course, mean some error in following a moving target. In the method described in this chapter such lags are automatically compensated for by adjustments in the coefficients of the other terms of the polynomial.

with respect to disposable parameters in $\bar{E}(\lambda)$ and a prescribed weighting function $W_0(t,\lambda)$. The lower limit of the integral is indicated as $-\infty$ in compliance with the physical impossibility of discriminating between relevant and irrelevant data, with fixed linear networks, except on the basis of age. The burden of discrimination must be relegated to the weighting function which must be a function only of the age $t-\lambda$. Under the ideal restriction that $W_0(t-\lambda)$ is identically zero when $t-\lambda > T$ or $\lambda < t-T$, the indicated lower limit of the integral is purely nominal.

As in Section 10.2, it is convenient to conduct the analysis in terms of the age variable $\tau = t - \lambda$ introduced there. If

$$\bar{F}(\tau) = \bar{E}(\lambda) \qquad F(\tau) = E(\lambda)$$

the integral to be minimized may be expressed in the form

$$\int_0^\infty [\bar{F}(\tau) - F(\tau)]^2 \, W_0(\tau)\, d\tau \,. \tag{1}$$

In accordance with the discussion of quasi-distortionless transmission networks in Section A.10, Appendix A, the smooth curve $\bar{E}(\lambda)$ should be a polynomial in λ. Hence $\bar{F}(\tau)$ should be a polynomial in τ. It will be more convenient, however, to express $\bar{F}(\tau)$ formally as a linear combination of polynomials in τ which may be orthogonalized. Hence, let

$$\bar{F}(\tau) = V_0 + V_1 \cdot G_1(\tau) + V_2 \cdot G_2(\tau) + \cdots + V_n \cdot G_n(\tau) \tag{2}$$

where $G_m(\tau)$ is an mth degree polynomial in τ.

Let $W_0(\tau)$ be normalized in the sense that

$$\int_0^\infty W_0(\tau)\, d\tau = 1$$

and the $G_m(\tau)$ be orthogonalized with respect to the weighting function $W_0(\tau)$ in the sense that

$$\int_0^\infty G_l(\tau)\, G_m(\tau)\, W_0(\tau)\, d\tau = 0 \quad \text{if } l \neq m$$
$$= \frac{1}{k_m} \quad \text{if } l = m$$

$(G_0 = 1,\ k_0 = 1)$.

The integral (1) is then a minimum with respect to the V_m's in (2) if

$$V_m = k_m \int_0^\infty F(\tau) \cdot G_m(\tau) \cdot W_0(\tau)\, d\tau \,. \tag{3}$$

In terms of the forward time λ, (2) and (3) reduce to

$$\bar{E}(\lambda) = V_0(t) + V_1(t) \cdot G_1(t-\lambda) + V_2(t) \cdot G_2(t-\lambda) + \cdots + V_n(t) \cdot G_n(t-\lambda) \tag{4}$$

where

$$V_m(t) = k_m \int_{-\infty}^t E(\lambda) \cdot G_m(t-\lambda) \cdot W_0(t-\lambda)\, d\lambda \,. \tag{5}$$

Expression (5) identifies the $V_m(t)$ as the responses to $E(\lambda)$ of fixed linear networks whose impulsive admittances are

$$W_m(\tau) = k_m G_m(\tau) \cdot W_0(\tau) \,. \tag{6}$$

By (4), the predicted value may be obtained by a linear combination of the responses of these networks, viz.,

$$\bar{E}(t+t_f) = V_0(t) + G_1(-t_f) \cdot V_1(t) + G_2(-t_f) \cdot V_2(t) + \cdots + G_n(-t_f) \cdot V_n(t) \,. \tag{7}$$

A schematic representation of an nth order smoothing and prediction circuit, based on (7), is shown in Figure 1, where the $G_m(-t_f)$ are represented as potentiometer factors dependent on the time of flight.

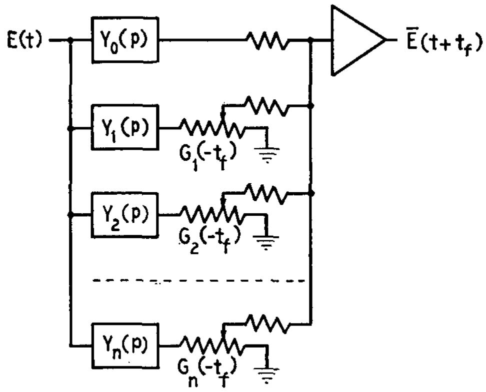


FIGURE 1. Schematic representation of nth order smoothing and prediction circuit.

Alternatively, (7) may be written

$$\bar{E}(t+t_f) = \bar{E}(t) + [G_1(-t_f) - G_1(0)] \cdot V_1(t) + \cdots + [G_n(-t_f) - G_n(0)] \cdot V_n(t) \tag{8}$$

where $\bar{E}(t)$ is then replaced by $E(t)$ when position data smoothing is to be omitted.

It is not necessary that the $G_m(\tau)$ polynomials be orthogonal. However, the circuit switching required to reduce or increase the order of the prediction is simplest when the $G_m(\tau)$ polynomials are orthogonal. Orthogonal polynomials corresponding to any prescribed weighting function $W_0(\tau)$ are readily derived by well-known methods.

The weighting function $W_0(\tau)$ may be determined by either of the methods described in Appendix B as the best weighting function for smoothing position data, under prescribed tracking error characteristics. Then the best impulsive admittances $W_m(\tau)$ for a smoothing and prediction circuit, are prescribed by (6).

The relationship (6) shows that if the prescribed weighting function $W_0(\tau)$ satisfies the formal requirements for physical realizability, so will all of the impulsive admittances $W_m(\tau)$. Of the standard sets of orthogonal polynomials those of Laguerre appear to be the best adapted to physical realization. The Laguerre polynomials $L_n^{(a)}(\tau)$ are orthogonal in $0 \leq \tau < \infty$ with the weighting function $\tau^a e^{-\tau}$. However, such a weighting function is, in general, very unsatisfactory from the practical point of view of settling characteristics.

It is possible of course to approximate any prescribed weighting function $W_0(\tau)$ as closely as may be desired in a physically realizable form, derive a set of orthogonal polynomials based on the approximate form, and determine the impulsive admittances $W_m(\tau)$ from (6). However, such a procedure leads to complexities of network configuration which increase very rapidly with the index m. This increasing complexity is hardly justifiable in practice.

From the foregoing considerations, it appears that the most practical procedure is to derive all of the impulsive admittances $W_m(\tau)$ without regard to physical realizability, approximate them independently in physically realizable forms of independently prescribed complexities, and modify or redetermine the potentiometer factors in accordance with the discussion in Section A.10, Appendix A.

11.2 WEIGHTING FUNCTIONS FOR DERIVATIVES

The impulsive admittances defined by (6) for $m > 0$ may not be regarded as weighting functions even though the response of the corresponding networks to $E(\lambda)$ is, by (5)

$$V_m(t) = \int_0^\infty E(t-\tau) \cdot W_m(\tau) \cdot d\tau ,$$

because, with the exception of $W_0(\tau)$, the $W_m(\tau)$, as will presently be seen, cannot be normalized. The term weighting function is reserved for the functions defined by (11) below.

Since τ^r is a linear combination of the $G_s(\tau)$ where $s = 0, 1, \cdots, r$, it is obvious from (6) that

$$\int_0^\infty \tau^r W_m(\tau)\, d\tau = 0$$

when $r < m$.

In particular

$$\int_0^\infty W_m(\tau)\, d\tau = 0$$

when $m > 0$.

Since the transmission function $Y_m(p)$ of a network is the Laplace transform of its impulsive admittance (see Section A.3), we have

$$\begin{aligned} Y_m(p) &= \int_0^\infty W_m(\tau)\, e^{-p\tau}\, d\tau \\ &= \sum_{r=0}^{\infty} \frac{(-p)^r}{r!} \int_0^\infty \tau^r W_m(\tau)\, d\tau . \end{aligned} \tag{9}$$

The first m terms in this series vanish. Hence $Y_m(p)$ will be of the form

$$Y_m(p) = p^m y_m(p) \tag{10}$$

where $y_m(0) \neq 0$. This permits us to regard the network whose impulsive admittance is $W_m(\tau)$ as an instantaneous mth order differentiator, corresponding to the factor p^m in (10), in tandem with a purely smoothing network whose transmission function is $y_m(p)$.

It is convenient to associate a weighting function $w_m(\tau)$ with the purely smoothing network whose transmission function is $y_m(p)$. Dividing (10) through by p^m the resulting operational equation may be interpreted (see Section A.5) to mean that the weighting function $w_m(\tau)$ is the m-fold integral of the impulsive admittance $W_m(\tau)$ between the limits 0 and τ. This is expressed by

$$w_m(\tau) = \int_0^\tau \cdots \int_0^\tau W_m(\tau) \cdot (d\tau)^m . \tag{11}$$

By a relationship similar to (9) between $y_m(p)$ and $w_m(\tau)$, it follows from $y_m(0) \neq 0$ that

$$\int_0^\infty w_m(\tau)\, d\tau \neq 0 .$$

Hence the $w_m(\tau)$ may be normalized in the sense that

$$\int_0^\infty w_m(\tau)\, d\tau = 1$$

for all values of m. However, this may be done in general only if the $G_m(\tau)$ polynomials are not normalized in the sense that $k_m = 1$ for any value of $m > 0$. It is in fact readily shown that the coefficient of τ^m in $G_m(\tau)$ must be the same as that of τ^m in $e^{-\tau}$.

11.3 LEGENDRE POLYNOMIALS

The Legendre polynomials $P_m(x)$ are orthogonal with respect to the range $-1 \leq x \leq 1$ and uniform weighting. In other words, the polynomials $P_m(2\tau - 1)$ are orthogonal with respect to the range $0 \leq \tau \leq \infty$ and the weighting function[c]

$$\begin{aligned} W_0(\tau) &= 1 \quad \text{when } 0 \leq \tau \leq 1 \\ &= 0 \quad \text{when } \tau > 1 \, . \end{aligned}$$

It is known from Section 10.4 that this form for the weighting function $W_0(\tau)$ is best in case the tracking errors are flat random noise. In the integral (1) to be minimized, the $G_m(\tau)$ polynomials should then be

$$G_m(\tau) = (-)^m \frac{m!}{(2m)!} P_m(2\tau - 1) \, .$$

The first few of these are tabulated below.

m	$G_m(\tau)$
0	1
1	$\frac{1}{2} - \tau$
2	$\frac{1}{12} - \frac{\tau}{2} + \frac{\tau^2}{2}$
3	$\frac{1}{120} - \frac{\tau}{10} + \frac{\tau^2}{4} - \frac{\tau^3}{6}$

With the help of the formula

$$\int_{-1}^1 [P_m(x)]^2\, dx = \frac{2}{2m+1}$$

[c] The unit of time being equal to the nominal smoothing time.

it is readily determined that

$$\frac{1}{k_m} = \int_0^\infty [G_m(\tau)]^2 W_0(\tau)\, d\tau = \frac{(m!)^2}{(2m)!\,(2m+1)!} \, .$$

Then, by (6)

$$\begin{aligned} W_m(\tau) &= (-)^m \frac{(2m+1)!}{m!} P_m(2\tau - 1) \quad 0 \leq \tau \leq 1 \\ &= 0 \quad \tau > 1 \, . \end{aligned}$$

Substituting this in turn into (11) and making use of Rodrigues' formula

$$P_m(x) = \frac{(-)^m}{2^m m!} \frac{d^m}{dx^m} (1 - x^2)^m$$

or

$$P_m(2\tau - 1) = \frac{(-)^m}{m!} \frac{d^m}{d\tau^m} [\tau(1-\tau)]^m$$

it is finally found that

$$\begin{aligned} w_m(\tau) &= \frac{(2m+1)!}{(m!)^2} [\tau(1-\tau)]^m \quad 0 \leq \tau \leq 1 \\ &= 0 \quad \tau > 1. \end{aligned} \tag{12}$$

By a relationship of the form of (9) the transmission functions $y_m(p)$ corresponding to the weighting functions $w_m(\tau)$ may be determined. The first three are

$$y_0(p) = \frac{1 - e^{-p}}{p}$$

$$y_1(p) = \frac{6}{p^3} [(p-2) + (p+2)e^{-p}]$$

$$y_2(p) = \frac{60}{p^5} [(p^2 - 6p + 12) - (p^2 + 6p + 12)e^{-p}].$$

These may be written in the form

$$y_m(p) = Q_m(\omega) \cdot e^{-i\omega/2} \tag{13}$$

where

$$Q_0(\omega) = \frac{\sin x}{x} \quad \left(x = \frac{\omega}{2}\right)$$

$$Q_1(\omega) = \frac{3 \sin x - x \cos x}{x^3}$$

$$Q_2(\omega) = 15 \frac{(3 - x^2) \sin x - 3x \cos x}{x^5} \tag{14}$$

or in the infinite power-series form

$$y_0(p) = \sum_{n=0}^{\infty} \frac{(-p)^n}{(n+1)!}$$

$$y_1(p) = 6 \sum_{n=0}^{\infty} \frac{n+1}{(n+3)!} (-p)^n$$

$$y_2(p) = 60 \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{(n+5)!} (-p)^n . \quad (15)$$

Methods for obtaining physically realizable approximations to the weighting functions $w_m(\tau)$ or impulsive admittances $W_m(\tau)$, based upon the Q functions (14) and the series expansions (15) are described in Chapter 12.

Chapter 12

PHYSICAL REALIZATION OF DATA-SMOOTHING FUNCTIONS

THIS CHAPTER will be devoted to a brief review of some of the methods and techniques which have been used in the physical realization of data-smoothing or weighting functions. The first two sections will be devoted to methods for determining physically realizable approximations to a desired weighting function. The third section takes up the use of feedback amplifiers and servomechanisms in order to avoid the use of coils of generally fantastic sizes. The final section takes up the design of resistance-capacitance networks.

Methods of deriving physically realizable approximations of best weighting functions may be divided into two classes, which may be called, for convenience, t-methods and p-methods. The t-methods are those in which a prescribed best weighting function $W(t)$ is approximated directly by a function $W_a(t)$ of realizable form, viz., a sum of decaying exponential terms and exponentially decaying sinusoidal terms. However, the t-methods are most useful when the approximation is restricted to a sum only of exponential terms. According to the discussion in Section A.9, Appendix A, such a restriction corresponds physically to passive RC transmission networks. A t-method was used by Phillips and Weiss in the reference quoted in Section 10.3 to obtain an approximation with one decaying exponential term and one exponentially decaying sinusoidal term. However, this method rapidly becomes unwieldy as the number of terms is increased.

The p-methods are those in which the approximation is derived indirectly from the transmission function $Y(p)$ corresponding to $W(t)$. A rational function $Y_a(p)$ approximating $Y(p)$ is first determined. If it is realizable, and it usually is, then $W_a(t) = L^{-1}[Y_a(p)]$. In general, $Y_a(p)$ will have complex poles and, therefore, $W_a(t)$ will have exponentially decaying sinusoids as well as simple exponentials. This gives the p-methods a considerable advantage over the t-methods in more efficient use of network elements. The fact that this generally calls for impractical element values in passive RLC networks is not serious. As shown in Section 12.3, the use of coils may be avoided entirely by the use of feedback amplifiers.

12.1 t-METHODS

To describe the t-method,[a] let

$$W_a(t) = A_1e^{-\alpha_1 t} + A_2e^{-\alpha_2 t} + \cdots + A_n e^{-\alpha_n t} \tag{1}$$

where the α's are prescribed and the A's are to be determined. Two considerations are involved in the determination of the A's. The first consideration is based on the relationship between the continuity conditions at $t = 0$ and the ultimate slope of the loss characteristic as expressed in the theorem in Section A.8. Accordingly, a number of relations of the type

$$\begin{aligned} A_1 + A_2 + \cdots + A_n &= 0 \\ \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_1^r A_1 + \alpha_2^r A_2 + \cdots + \alpha_n^r A_n &= 0 \quad r < n-1 \end{aligned} \tag{2}$$

must be satisfied. This leaves $n - r - 1$ of the A's for the second consideration.

The second consideration concerns the manner in which the approximation in the range $t > 0$ is to be made. The approximation may, for example, be required to pass through $n - r - 1$ points on $W(t)$ or, the first $n - r - 1$ moments of the approximation may be required to be equal to the corresponding moments of $W(t)$. The latter is expressed by relations of the type

$$\frac{A_1}{\alpha_1^s} + \frac{A_2}{\alpha_2^s} + \cdots + \frac{A_n}{\alpha_n^s} = \frac{1}{(s-1)!}\int_0^\infty W(t)\, t^{s-1}\, dt$$
$$s = 1, 2, \cdots, n - r - 1 \tag{3}$$

Foster's investigations were concerned only with the parabolic weighting function (4) Chapter 10, so that only the first of (2) was involved. Numerical studies led to the belief that, with a given number of α's, the best approximation was to be had from the case in

[a] The t-method is principally due to R. M. Foster.

which all of the α's are equal. Hence the natural center of attention was the special form

$$W_a(t) = (A_1t + A_2t^2 + \cdots + A_{n-1}t^{n-1})e^{-\alpha t}. \quad (4)$$

At large values of t this expression reduces approximately to the last term, and if it is assumed that $A_{n-1} \doteq 1$, the settling condition fixes α to at least a first approximation. The rest of the work of approximating the parabola is then equivalent to a problem in polynomial approximation. Once the A's are determined, a better value of α can be found from the settling condition, and the process gone through again.

If the α's are only approximately equal, the approximation will still behave approximately like (4) with an average value used for α. The difficulty with equal or nearly equal α's is that it leads to networks with extreme element values. In order to secure satisfactory element values, it is generally necessary to depart substantially from the condition of equal α's. This results in some, but not a large, loss of efficiency in approximating the parabola. Foster recommends that the α's be chosen as a geometric series, with their geometric mean more or less around the equivalent point for equal α's. With four α's he suggests that the constant ratio in the series may be 3:2, whereas with only two α's the ratio should be raised to 2:1. These are, however, only rough values and obviously depend on individual opinion of what constitutes an unreasonable element value.

As a matter of experience, it turns out that the characteristic first obtained usually has a rather long and slowly decaying tail, as shown in Figure 1. This, of course, is equivalent to a

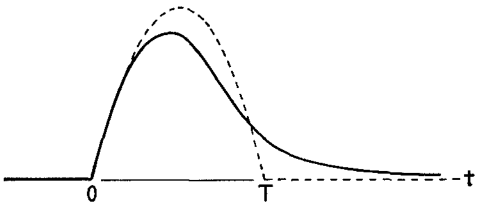


FIGURE 1. Approximation to parabolic weighting function, showing poor settling characteristic.

correspondingly long "settling time," or time before a useful prediction can be made. In practice, therefore, after the preliminary design has been found, adjustments are made to bring the tail of the curve under control, partly by modifying the values of the A's slightly, and partly by contracting the time scale to bring the part of the tail which remains appreciable within the allowable settling time limits. This leads to the somewhat lopsided match to the parabola shown in Figure 2.

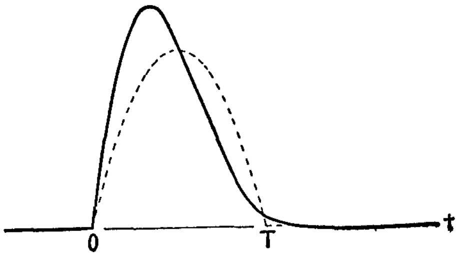


FIGURE 2. Approximation to parabolic weighting function, showing better settling characteristic.

A method of bringing the tail of the curve under control[h] is to minimize the expression

$$\int_{t=T}^{\infty} [W_a(t)]^2\, dt = \sum_{l,m=1}^{n}\sum C_{lm}\, A_l\, A_m \quad (5)$$

where

$$C_{lm} = \frac{e^{-(\alpha_l+\alpha_m)T}}{\alpha_l + \alpha_m}$$

under the restrictions (2) and all but the last of (3).

The t-method used by Phillips and Weiss is based on a 3-term approximation of the form (1) in which one α is real while the other two may be conjugate complex. The α's are not prescribed, so that there are six parameters to be determined. Four restrictions are imposed, viz., the first of (2), the first of (3), a restriction on the value of the tail area, viz.,

$$\int_T^{\infty} W_a(t)\, dt = \sum_{l=1}^{3} \frac{A_l\, e^{-\alpha_l T}}{\alpha_l},$$

and the cross-over condition

$$W_a(T) = 0.$$

Finally, the transmitted noise power, which, under the assumption of flat random noise associated with the position data, takes the form (see Section 10.4)

$$\int_0^{\infty} [W_a(t)]^2\, dt$$

is minimized with respect to the two remaining parameters by numerical methods.

[h] Used by R. F. Wick.

12.2 *p*-METHODS

Three *p*-methods have been used. These will be described in chronological order.

The first *p*-method is one which was used by R. L. Dietzold in exploiting the use of feedback amplifiers to secure the advantages of approximations with complex exponentials. The transmission function $Y(p)$ corresponding to the best weighting function $W(t)$ is first formulated. The loss characteristic, $-20 \log_{10} |Y(i\omega)|$, is next computed and plotted against the frequency on a logarithmic scale. Then standard equalizer design techniques are employed to approximate the loss characteristic, keeping in mind that the transmission loss in the feedback network of a feedback amplifier becomes a transmission gain for the circuit as a whole (see Section 12.3).

The second *p*-method is merely a more complete analytic formulation of the first, thereby avoiding the necessity for employing equalizer design techniques. It depends upon the possibility of expressing the transmission function corresponding to the best weighting function, in the form of equation (13) Chapter 11, which is associated with the symmetry of the weighting function, as shown in Section A.7. The method is based upon the determination of the envelope of the *Q*-function. The *Q*-function is first differentiated in order to obtain the equation which determines the values of ω at which the maxima and minima occur. This transcendental equation is not solved but is used to eliminate the trigonometric functions in the expression of the *Q*-function. The resulting expression, which is an irrational function of ω^2, is then squared in order to make it a rational function of ω^2. The substitution $p^2 = -\omega^2$ is made and the expression is then resolved into two factors of which one contains all the poles with negative real parts while the other contains all the poles with positive real parts, the two factors being conjugate complex when $p = i\omega$. The first factor is then taken as an approximation of the desired transmission function. Applying the method to the desired transmission functions defined by (13) and (14) of Chapter 11, we get

$$y_0(p) \doteq \frac{2}{2+p}$$

$$y_1(p) \doteq \frac{12}{12+6p+p^2}$$

$$y_2(p) \doteq \frac{120}{120+60p+12p^2+p^3}\,. \tag{6}$$

This last is the basis for the design of a position and rate smoothing circuit for a proposed computor for controlling bombers from the ground.[11,12] This design is described briefly in Chapter 13.

The third *p*-method is based upon the ascending power-series expansion of the transmission function corresponding to the best weighting function. Examples of such power series are given by (15) of Chapter 11. The method of approximation is one which is credited to Padè in O. Perron's "Kettenbrüchen."[18] If the discussion in Section A.8 is referred to, it will be seen to be also a method of moments.

The method consists in determining the coefficients in a rational function of the form

$$\frac{1 + a_1 p + a_2 p^2 + \cdots + a_m p^m}{1 + b_1 p + b_2 p^2 + \cdots + b_n p^n} \tag{7}$$

so that the ascending power-series expansion of the rational function will agree with that of the best transmission function, term for term up to and including p^{m+n}. If the series for the best transmission function is

$$1 + c_1 p + c_2 p^2 + \cdots + c_{m+n} p^{m+n} + \ldots \tag{8}$$

the equations which determine the coefficients in (7) are obtained by equating coefficients of corresponding powers of p, up to and including the $(m+n)$th, in

$$(1 + b_1 p + \cdots + b_n p^n)\,(1 + c_1 p + \cdots + c_{m+n} p^{m+n})$$

and

$$1 + a_1 p + \cdots + a_m p^m.$$

The last n equations will be homogeneous in the b's and c's.

It has been expedient in some cases to omit the last few of the $(m+n)$ equations in order to have some control over the number of real roots and poles and the number of conjugate pairs of complex roots and poles in the resulting rational function.

In the assumed rational expression (7) the difference $n - m$ should be chosen so that the ultimate slope of the loss characteristic will be the same as for the best transmission function. According to the theorem in Section A.8, if $W(t)$ behaves like t^r as $t \to 0$, we should take $n - m = r + 1$. As a matter of experience the rational expression has invariably turned out to be physically realizable whenever this "rule" was followed. Frequently, however, the rational expression has turned out to be physically realizable under small departures from the rule.

Examples of this method are given in Chapter 13.

12.3 USE OF FEEDBACK AMPLIFIERS AND SERVOMECHANISMS

In this section we shall describe the use of feedback amplifiers and servomechanisms to obtain desired transmission functions. For complete discussions of the most recent technical advances in the analysis and design of feedback amplifiers and servomechanisms the reader should consult some of the modern literature on these subjects.[2,3,5,15,16,17]

Let us assume that we have two networks whose transmission functions are $Y_1(p)$ and $Y_2(p)$, respectively, as shown in Figure 3. For

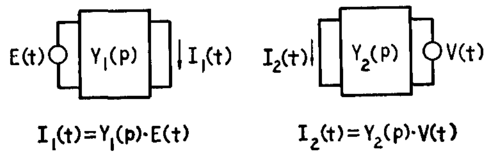


FIGURE 3. Schematic representation of networks intended for feedback circuit application.

a signal $E(t)$ applied to the first network the short-circuit output current is $I_1(t) = Y_1(p) \cdot E(t)$. For a signal $V(t)$ applied to the second network the short-circuit output current is

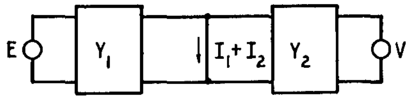


FIGURE 4. First step in combining networks.

$I_2(t) = Y_1(p) \cdot V(t)$. With the networks sharing a common short-circuiting conductor as shown in Figure 4, the current through the conductor is $I_1 + I_2$. If the source which develops the voltage $V(t)$ across the input terminals of the second network were in fact under the control of the current through the conductor, as shown schematically in Figure 5, in such a manner

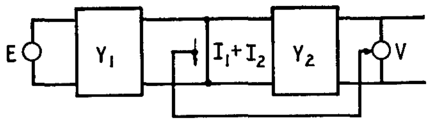


FIGURE 5. Output voltage controlled by short-circuit current across intermediate terminals.

that it had to develop that voltage $V(t)$ which reduces the current in the conductor to zero, then

$$Y_1(p) \cdot E(t) + Y_2(p) \cdot V(t) = 0 .$$

Hence, the transmission function (now a voltage-voltage ratio) of the arrangement shown in Figure 5 must be

$$Y(p) = -\frac{Y_1(p)}{Y_2(p)} . \tag{9}$$

This relationship provides a method of obtaining transmission functions with complex poles without the requirement of coils.[c] The complex roots of $Y(p)$, must be assigned to the numerator of $Y_1(p)$, and the complex poles of $Y(p)$ to the numerator of $Y_2(p)$. Aside from this, the other roots and poles of $Y(p)$ may be assigned in any way which is favorable to good design practice. Redundant factors may be introduced if they are desirable, as is done in the examples described in Sections 13.1.5 and 13.3.

The source of the voltage $V(t)$ in Figure 5 does not have to be controlled by the current through the short-circuiting conductor. Since the current through any short circuit must be zero if the voltage across the short-circuited terminals is zero before the short circuit is connected across them, the source of the voltage $V(t)$ may just as well be controlled by the open-circuit voltage, as shown in Figure 6. It is clear that the source of the voltage $V(t)$ is ideally an infinite gain amplifier. It is not necessary, however, that the amplifier have ideally unilateral transmission and infinite input and output impedances, since departures from these ideal characteristics may be compensated for in the design of the feedback network.

The simple result expressed by (9) may be readily modified to take account of the finite

[c] This observation was first made by R. L. Dietzold.

gain of a physical amplifier. The modification will be expressed as an extra factor which corresponds to the "$\mu\beta$ effect" or "$\mu\beta$ error"[15e] commonly encountered in the theory and design of feedback amplifiers.

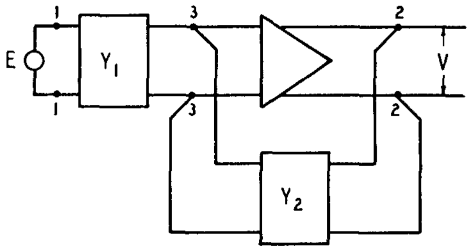


FIGURE 6. Output voltage controlled by open-circuit voltage across intermediate terminals.

The exact transmission function of the circuit shown in Figure 6 is most simply expressed in terms of the following quantities:

$Y_1(p)$ = current through a short across terminal-pair No. 3, per unit emf applied across terminal-pair No. 1.

$Y_2(p)$ = current through a short across terminal-pair No. 3, per unit emf applied across terminal-pair No. 2.

$Z_2(p)$ = impedance between terminal-pair No. 2, with terminal-pair No. 3 shorted.

$Z_3(p)$ = impedance between terminal-pair No. 3, with amplifier dead, terminal-pair No. 1 shorted, and terminal-pair No. 2 open.

$G(p)$ = transadmittance of amplifier.

Then

$$Y = -\frac{Y_1}{Y_2}\,\frac{1+\dfrac{Y_2}{G}}{1-\dfrac{1}{GY_2Z_2Z_3}}\,. \tag{10}$$

The quantity $GY_2Z_2Z_3$ is the $\mu\beta$ of the circuit. The quantity $Y_1Y_2Z_2Z_3$ to which Y reduces when $G = 0$ represents the direct transmission of the circuit.

The active impedance across terminal-pair No. 2 is

$$Z_{2A} = \frac{Z_{2P}}{1 - GY_2Z_2Z_3} \tag{11}$$

where

$$Z_{2P} = Z_2(1 + Y_2^2Z_2Z_3)\,. \tag{12}$$

Z_{2P} is the passive impedance across terminal-pair No. 2. It differs from Z_2 in that terminal-pair No. 3 is open.

The exact expression (10) of the transmission function is useful chiefly as a check on the simpler but approximate expression (9). It is in general quite practicable to make the transadmittance or transconductance G of the amplifier large enough so that the $\mu\beta$ effect may be neglected.

In accordance with the sense in which the term "servomechanism" is used by MacColl,[4] a feedback circuit, such as that shown in Figure 6, is a servomechanism — more specifically, an electronic servomechanism — since it operates on the ideal principle of maintaining zero voltage across the terminal-pair No. 3. An electromechanical counterpart of the circuit shown in Figure 6 is shown in Figure 7. These

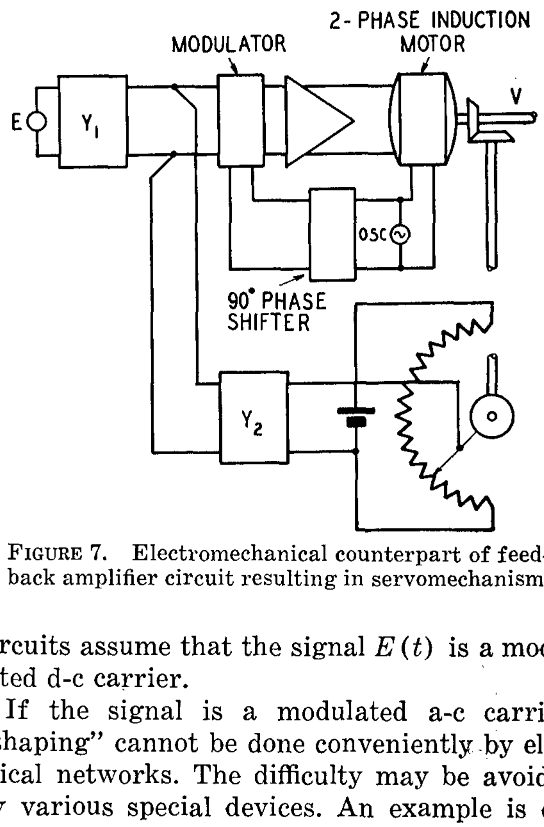


FIGURE 7. Electromechanical counterpart of feedback amplifier circuit resulting in servomechanism.

circuits assume that the signal $E(t)$ is a modulated d-c carrier.

If the signal is a modulated a-c carrier, "shaping" cannot be done conveniently by electrical networks. The difficulty may be avoided by various special devices. An example is described and illustrated in Section 13.4.

12.4 DESIGN OF *RC* NETWORKS

In this section we will describe and illustrate two general methods of designing *RC* networks. The first is most useful when the transmission function is finite and not zero at zero frequency; the second, when the transmission function is zero at zero frequency. The case of a transmission function with a pole at zero frequency will not be considered, since it is covered by the methods described in the preceding section, in conjunction with the methods described below.

Let

$$Y(p) = \frac{a_0 + a_1 p + \cdots + a_{n+1}p^{n+1}}{1 + b_1 p + \cdots + b_n p^n} \quad (a_0 > 0) \quad (13)$$

with simple, real, negative poles. Dividing by p, expanding into partial fractions and multiplying through by p, we get

$$Y(p) = \frac{pa_{n+1}}{b_n} + \left(a_0 + \frac{pA_1}{p+\alpha_1} + \frac{pA_2}{p+\alpha_2} + \cdots\right) - \left(\frac{pB_1}{p+\beta_1} + \frac{pB_2}{p+\beta_2} + \cdots\right)$$

where the A's, B's, α's and β's are positive real quantities. The first term must be associated with those in the first parentheses if $a_{n+1} > 0$, with those in the second parentheses if $a_{n+1} < 0$. The transmission function is now in the form

$$Y(p) = Y_A(p) - Y_B(p) \quad (14)$$

where $Y_A(p)$ and $Y_B(p)$ are physically realizable driving-point admittances of RC type. Each term of the form $pA/(p+\alpha)$ is the admittance of the two-terminal, two-element network

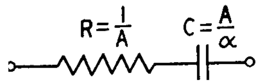


FIGURE 8. Simple RC network.

shown in Figure 8. Each term in (14) therefore represents a parallel combination of two-element networks of the type shown in Figure 8 and a conductance a_0 in the case of $Y_A(p)$,

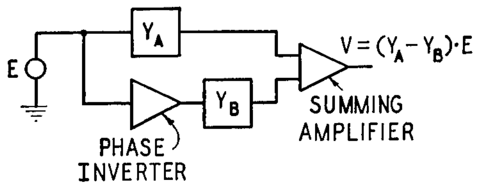


FIGURE 9. Method of realizing RC transmission functions, requiring phase inverter.

and a capacitance $|a_{n+1}|/b_n$ in the case of either $Y_A(p)$ or $Y_B(p)$. By well-known methods these two-terminal networks may be transformed into a variety of other configurations.

The transmission function (14) may be realized in the arrangement shown in Figure 9 or in that shown in Figure 10. The latter is a lattice network which is suitable only in a

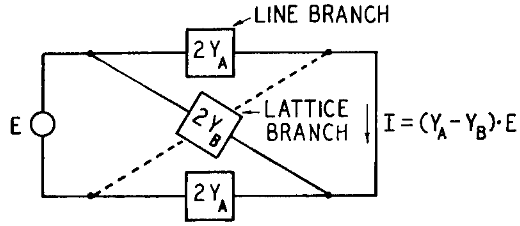


FIGURE 10. Lattice prototype for passive networks with RC transmission characteristics.

balanced-to-ground circuit. To obtain an unbalanced passive equivalent of this network we may resort to steps which will be described later in this section.

The second general method of designing RC networks is most useful when

$$Y(p) = p\,\frac{a_0 + a_1 p + \cdots + a_n p^n}{1 + b_1 p + \cdots + b_n p^n} \quad (a_0 > 0) \quad (15)$$

with simple, real, negative poles. Now, if the lattice in Figure 10 were driven from an infinite-impedance source of current I_0, the output current would be

$$I = \frac{1 - \dfrac{Y_B}{Y_A}}{1 + \dfrac{Y_B}{Y_A}} I_0 .$$

If, furthermore,

$$\frac{Y_B}{Y_A} = \frac{k - \dfrac{Y}{p}}{k + \dfrac{Y}{p}} \quad (16)$$

then

$$I = \frac{Y}{kp} I_0 .$$

Taking it for granted for the moment that the lattice can be transformed as shown schematically in Figure 11, we may then discard the condenser across the output terminals and, by Thévenin's theorem,[15f] we may replace the condenser across the input terminals and the infinite-impedance current source by a series condenser and a zero-impedance voltage source. The result is shown in Figure 12. Since

$I_0 = pC_0E$ we now have

$$I = \frac{C_0}{k} Y \cdot E$$

which is the desired result, to a constant factor.

The factor k should in general be taken as small as possible subject to the requirement that all the roots and poles of (16) be simple,

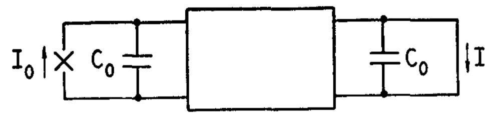


FIGURE 11. Step in transformation of networks with zero transmission at zero frequency.

real, and negative. It can always be taken large enough to fulfill this requirement. A suitable value may be easily chosen by inspection of a plot of $Y(p)/p$ for negative real values of p.

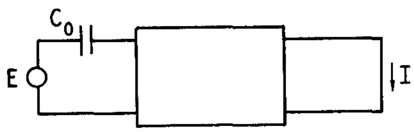


FIGURE 12. Final step in transformation of networks with zero transmission at zero frequency.

The numerator and denominator of (16) are of equal degree and therefore contain the same number of linear factors. These factors may be assigned to Y_A or to Y_B arbitrarily except that Y_A and Y_B must be physically realizable driving-point admittance functions which behave ultimately like condensers as the frequency increases indefinitely; that is, roots and poles must alternate and there must be a simple pole at infinity.

There are five kinds of steps which may be taken to transform a lattice into an unbalanced form. These steps are based upon Bartlett's bisection theorem,[14] and may be taken in any order and as often as necessary. Each of them will now be described as it would be applied directly to Figure 10. In the following diagrams a lattice enclosed in a rectangle means an unbalanced network whose configuration may not be known yet, but whose lattice prototype is as indicated.

1. Shunt network pulled out of both branches: shown in Figure 13.

2. Shunt network pulled out of the line branch only: shown in Figure 14.

3. Series network pulled out of both branches: shown in Figure 15.[c]

4. Series network pulled out of the lattice branch only: shown in Figure 16.[c]

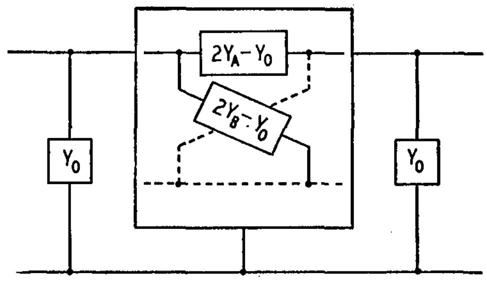


FIGURE 13. Step in transformation of lattice; shunt networks pulled out of both branches.

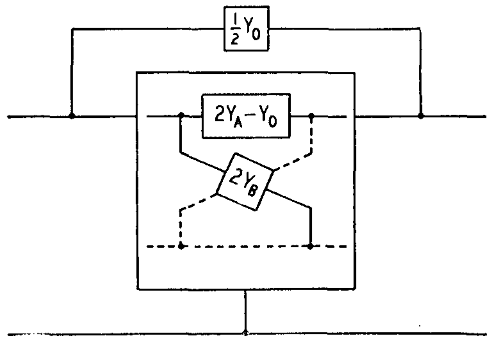


FIGURE 14. Step in transformation of lattice; shunt network pulled out of line branch only.

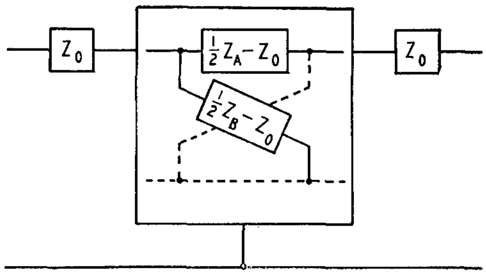


FIGURE 15. Step in transformation of lattice; series networks pulled out of both branches.

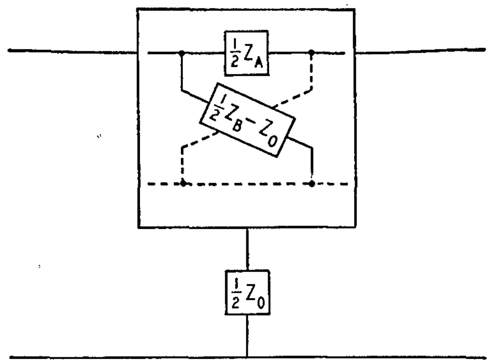


FIGURE 16. Step in transformation of lattice; series network pulled out of lattice branch only.

[c] Given in impedance form.

5. Breakdown into parallel lattices: a fairly obvious step which need not be illustrated.

As an example of (13) consider

$$Y(p) = \frac{a_0 + a_1 p + a_2 p^2}{1 + b_1 p}$$

where all the coefficients are positive. Since

$$Y(p) = \frac{pa_2}{b_1} + a_0 - \frac{(a_0 b_1^2 - a_1 b_1 + a_2)p}{b_1^2\left(p + \frac{1}{b_1}\right)}$$

there is no problem if $a_1 > (a_2/b_1) + a_0 b_1$. But if $a_1 < (a_2/b_1) + a_0 b_1$ we have the problem of trans-

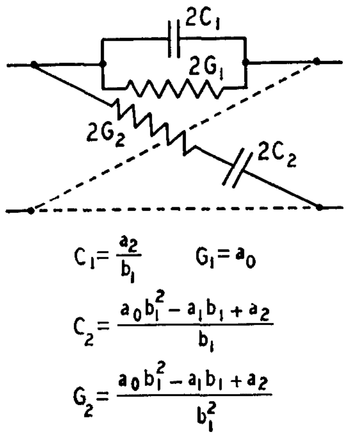


FIGURE 17. Illustrative lattice prototype.

forming the lattice in Figure 17. We can apply steps 2 and 4 immediately, but find that the residual lattice cannot be transformed unless $a_1 > (a_2/b_1)$. Under this additional restriction we can apply step 3 obtaining finally the network shown in Figure 18.

As an example of (15) consider

$$Y(p) = p\,\frac{1 + 12p}{1 + 24p + 64p^2}\,.$$

Taking $k = 1$ (the smallest value which may be assigned), we get

$$\frac{Y_B}{Y_A} = \frac{2p(3 + 16p)}{(1 + 2p)\,(1 + 16p)}\,.$$

One way of choosing Y_A and Y_B is

$$Y_A = \frac{(1 + 2p)\,(1 + 16p)}{2(3 + 16p)} \qquad Y_B = p\,.$$

This leads finally to the network shown in Figure 19. Such a simple network is possible of course because $Y(p)$ happens to satisfy the requirements of a physically realizable driving-point admittance function. However, another way of choosing Y_A and Y_B is

$$Y_A = \frac{1 + 2p}{2} \qquad Y_B = \frac{p(3 + 16p)}{1 + 16p}$$

This leads to the network shown in Figure 20.

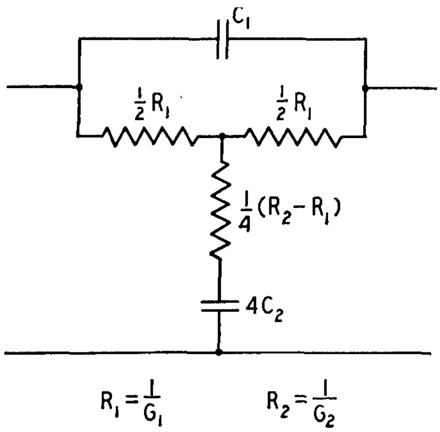


FIGURE 18. Unbalanced equivalent of illustrative lattice prototype when $a_2/b_1 < a_1 < (a_2/b_1) + a_0 b_1$.

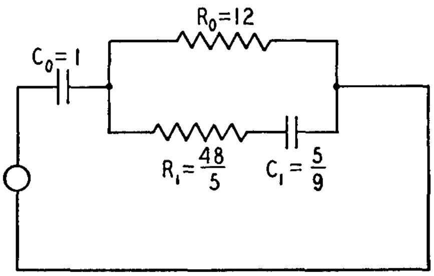


FIGURE 19. RC network with zero transmission at zero frequency.

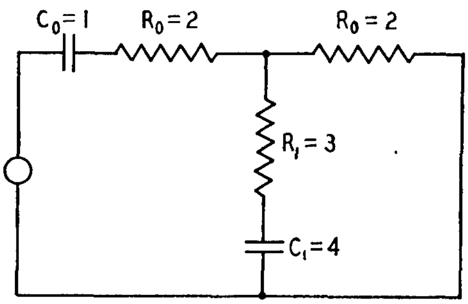


FIGURE 20. Another RC network with zero transmission at zero frequency.

Chapter 13

ILLUSTRATIVE DESIGNS AND PERFORMANCE ANALYSIS

THE ILLUSTRATIVE MATERIAL described in this chapter is taken from four practical applications.

1. Second-derivative circuit for the M9 antiaircraft director.
2. Position data smoother for the "close support plotting board," with delay correction for constant velocity aircraft.
3. Position and rate circuit for the "computer for controlling bombers from the ground," with optional delay correction of position data for constant-velocity aircraft.
4. Position and rate circuit using electromechanical servomechanisms.

The design and analytical procedure used in the first application has not heretofore been described in writing. Hence, considerably more space will be devoted to it than to the other three applications. The latter have been described in detail in reports.[10,12,13]

13.1 SECOND-DERIVATIVE CIRCUIT DESIGN

13.1.1 Realizable Approximation of Best Transmission Function

The best transmission function for the second-derivative circuit was taken to be

$$Y_2(p) = p^2 y_2(p)\,,$$

in the notation of Chapter 11. This assumes flat random noise in position data and, arbitrarily, 1-second smoothing and settling time. The series expansion of $y_2(p)$ is, according to expressions (15) of Chapter 11,

$$y_2(p) = 1 - \frac{1}{2}p + \frac{1}{7}p^2 - \frac{5}{168}p^3 + \frac{5}{1008}p^4 - \cdots .$$

The form of the rational approximation,

$$\bar{y}(p) = \frac{1}{1 + b_1 p + b_2 p^2 + b_3 p^3 + b_4 p^4},$$

was chosen for simplicity under the requirement that the transmission function $p^2\bar{y}(p)$ should cut off at the rate of 12 db per octave.[a] This requirement was set as a precaution against noise due to granularity of the coordinate-conversion potentiometers in the director.

Following the procedure outlined in Section 12.2 the following equations were obtained:

$$b_1 - \frac{1}{2} = 0$$

$$b_2 - \frac{1}{2}b_1 + \frac{1}{7} = 0$$

$$b_3 - \frac{1}{2}b_2 + \frac{1}{7}b_1 - \frac{5}{168} = 0$$

$$b_4 - \frac{1}{2}b_3 + \frac{1}{7}b_2 - \frac{5}{168}b_1 + \frac{5}{1008} = 0$$

whence

$$b_1 = \frac{1}{2},\ b_2 = \frac{3}{28},\ b_3 = \frac{1}{84},\ b_4 = \frac{1}{1764}\,.$$

Since

$$p^4 + 21p^3 + 189p^2 + 882p + 1764$$
$$= \left(p^2 + \frac{21 + \sqrt{21}}{2}p + 42\right)$$
$$\times \left(p^2 + \frac{21 - \sqrt{21}}{2}p + 42\right),$$

$\bar{y}_2(p)$ would have two conjugate pairs of complex poles, viz.,

$$p = -6.40 \pm i1.047, \quad -4.10 \pm i5.02,$$

of which one pair is very nearly real.

In order to simplify the circuit design, however, it was desirable to limit the number of complex poles to a single conjugate pair. This was accomplished by leaving b_4 arbitrary so that the denominator of $\bar{y}_2(p)$ was

$$1 + \frac{1}{2}p + \frac{3}{28}p^2 + \frac{1}{84}p^3 + b_4 p^4\,.$$

A value for b_4 which would make this expression vanish at two negative real values of p was found by plotting

$$1764 b_4 = \frac{21}{x^4}(x^3 - 9x^2 + 42x - 84)$$

[a] The design antedated the formulation of the $n - m = r + 1$ rule given in Section 12.2, according to which the best transmission function should have been taken as $p^2 y_3(p)$ in the notation of Chapter 11. However, no trouble was experienced in obtaining a physically realizable approximation, of the complexity assumed.

against x, as shown in Figure 1. The right-hand member is positive only in the range $x > 3.77$ and has a maximum of 0.982 at about $x = 6.63$.

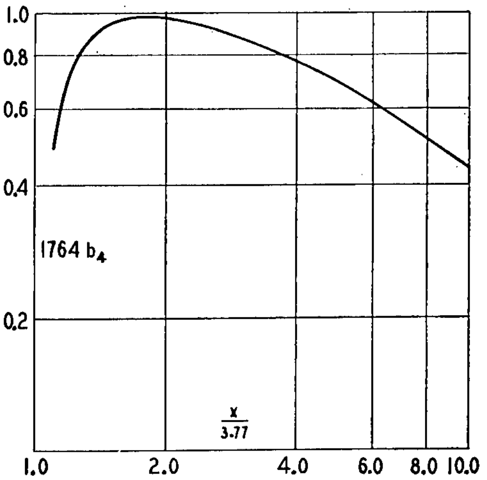


FIGURE 1. Graphical determination of b_4.

In order to obtain a substantial separation between the two real poles of $\bar{y}_2(p)$, the value $1764 b_4 = 0.5$ was chosen. The approximation

$$\bar{y}(p) = \frac{1}{1 + \frac{1}{2}p + \frac{3}{28}p^2 + \frac{1}{84}p^3 + \frac{1}{3528}p^4}$$

has poles at

$$p = -4.17391\ , \quad -31.72813\ , \quad -3.04898 \pm i\,4.16463\ .$$

The series expansion of $\bar{y}_2(p)$ agrees with that of $y_2(p)$ to four terms, the fifth term being $37/7056\, p^4$ instead of $5/1008\, p^4$. The difference in the fifth term is less than 6 per cent.

The realized approximation and the best weighting function are shown in Figure 3.

13.1.2 Transient Responses

The responses of the physical network whose transmission function is $p^2\bar{y}_2(p)$ are compared to those of the best network whose transmission function is $p^2 y_2(p)$, in Figures 2, 3, and 4. The signals for which (and the formulas by which) these responses were computed are tabulated below.

Figure	Signal $t \le 0$	Signal $t \ge 0$	Response formulas Realized	Response formulas Best
2	0	1	$L^{-1}[p\bar{y}_2(p)]$	$60t(1-2t)(1-t)$
3	0	t	$L^{-1}[\bar{y}_2(p)]$	$30[t(1-t)]^2$
4	0	$\frac{1}{2}t^2$	$L^{-1}\left[\frac{1}{p}\bar{y}_2(p)\right]$	$t^3(10-15t+6t^2)$

It has been noted that Figure 3 also represents the best and the realized weighting functions.

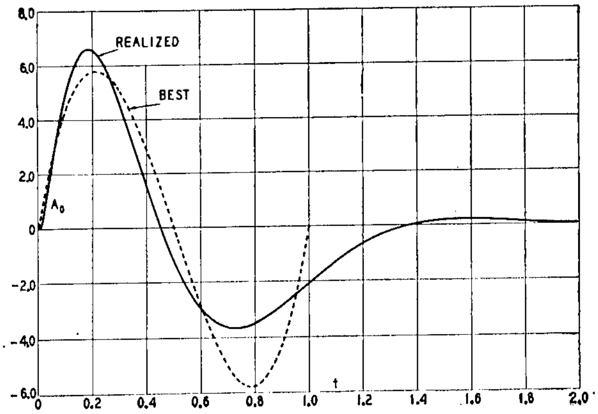


FIGURE 2. Responses to step function, viz., $E(t) = 1$ when $t > 0$.

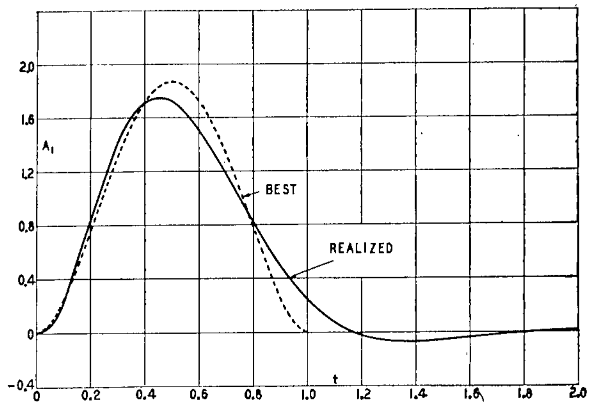


FIGURE 3. Responses to linear ramp function, viz., $E(t) = t$ when $t > 0$; second derivative smoothing functions.

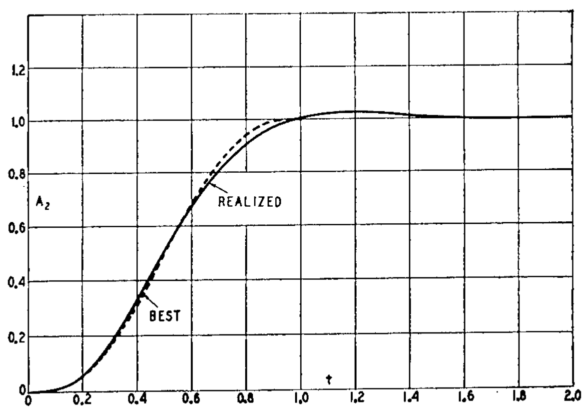


FIGURE 4. Responses to parabolic ramp function, viz., $E(t) = (\frac{1}{2})t^2$ when $t > 0$; second derivative settling characteristics.

If a signal of the form

$$E(t) = a_0 + a_1 t + \frac{1}{2} a_2 t^2$$

were to be applied suddenly to the second-derivative circuit at $t = 0$ the response would be

$$V(t) = \frac{a_0}{T^2} A_0 \left(\frac{t}{T}\right) + \frac{a_1}{T} A_1 \left(\frac{t}{T}\right) + a_2 A_2 \left(\frac{t}{T}\right) ,$$

where A_0, A_1, A_2 stand for the responses shown in Figures 2, 3, and 4, respectively, and where t is the time in seconds and T is the nominal smoothing time. The response $V(t)$ is the indicated acceleration of the target.

The sudden application of the instantaneous position and velocity components of the signal to the second-derivative circuit will give rise to some very serious consequences unless special measures are taken to mitigate them. To see this let it be assumed that $T = 20$ seconds and that the target is at such a range that $a_0 =$ 20,000 yards when the signal $E(t)$ is applied to the second-derivative circuit. Each unit of A_0 in the ordinate scale of Figure 2 then represents an indicated acceleration of 50 yd per sec². Referring to Figure 2 it is clear not only that the effective settling time will be several times the smoothing time but also that the indicated acceleration will go through exceedingly large maxima.

Exceedingly large transient responses are not peculiar to second-derivative circuits. They occur also in first-derivative circuits in linear prediction, where they are due entirely to the initial position term in the signal. In all cases they are reduced to harmless proportions by special arrangements of the circuits during the operation of slewing.

13.1.3 Effect of Tracking Errors on Accuracy of Prediction

The statistical effect of tracking errors on the accuracy of prediction is most readily determined from the power spectrum of the tracking errors and the transmission function of the prediction circuit.

Table 1 gives the values of the transmission function Y_1 of the first-derivative circuit in the M9 director, referred to a nominal smoothing time of 1 second,[b] and the transmission function Y_2 of the experimental second-derivative circuit design, also referred to a nominal smoothing time of 1 second. The transmission function of the linear prediction circuit with 10-second smoothing of first derivative is then

$$Y_l(p) = 1 + \frac{G_1 \cdot Y_1(10p)}{10}$$

TABLE 1*

$9f$	Y_1	i	Y_2	i
1	0.174	0.666	−0.454	0.165
2	0.651	1.166	−1.442	1.212
3	1.312	1.358	−2.014	3.527
4	1.943	1.203	−1.069	6.688
5	2.382	0.821	2.000	9.409
6	2.599	0.364	6.575	10.115
7	2.637	−0.067	10.893	8.220
8	2.558	−0.429	13.468	4.695
9	2.416	−0.711	14.096	0.953
10	2.242	−0.920	13.401	−2.092
11	2.062	−1.070	12.064	−4.320
12	1.885	−1.172	10.530	−5.777
13	1.720	−1.238	9.027	−6.704
14	1.566	−1.279	7.652	−7.169
15	1.429	−1.299	6.438	−7.398
16	1.305	−1.304	5.382	−7.446
17	1.194	−1.299	4.471	−7.374
18	1.096	−1.286	3.683	−7.221
19	1.004	−1.268	3.015	−7.025
20	0.926	−1.247	2.436	−6.795
22	0.790	−1.198	1.509	−6.292
24	0.683	−1.145	0.818	−5.780
26	0.593	−1.091	0.301	−5.287
28	0.518	−1.040	0.088	−4.826
30	0.457	−0.991	−0.380	−4.402
32	0.407	−0.945	−0.599	−4.016
34	0.364	−0.902	−0.762	−3.666
36	0.326	−0.862	−0.881	−3.348
38	0.296	−0.825	−0.967	−3.062
40	0.266	−0.790	−1.026	−2.800

* f is in c when smoothing time $T = 1$ sec. For T-second networks, values of $9f$ are multiples of $1/9T$ c, values of Y_1 should be divided by T, and values of Y_2 should be divided by T^2. The two networks may have different values of T.

while that of the quadratic prediction circuit with 20-second smoothing of second derivative is

$$Y_q(p) = Y_l(p) + \frac{G_2 \cdot Y_2(20p)}{400}$$

where G_1 and G_2 are determined in accordance with the discussion in Section A.10. Since

$$Y_1(p) = p(1 - 0.3724p + \cdots)$$
$$Y_2(p) = p^2(1 - \cdots)$$

we get

$$G_1 = t_f$$
$$G_2 = \frac{1}{2} t_f^2 + 3.724 t_f .$$

[b] $Y_1(p) = p\left(\frac{0.9494}{p+1.6} - \frac{8.677}{p+2.4} + \frac{34.74}{p+3.6} - \frac{27.01}{p+4.8}\right)$

Table 2 gives the values of $|Y_l(p)|^2$ and of $|Y_q(p)|^2$ for $t_f = 5, 10, 15, 20$ seconds. These are plotted in Figures 5, 6, 7, and 8.

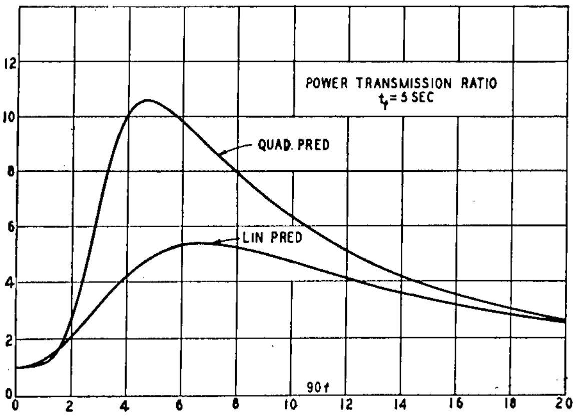


FIGURE 5. Power transmission ratio of linear and quadratic prediction circuits with 5-second prediction time.

The last column of Table 2 and Figure 9 give the power spectrum of a composite of the range and transverse errors in a typical run made with an experimental Mark VII radar. The power contained in the frequency range covered by the table accounts for 78 per cent of the total power, or an rms error of 15.8 yards out of 17.9 yards.

The rms error of prediction is the square root of the power transmitted by the prediction circuit. This is tabulated on the last line of Table 2 and in the smaller table following.

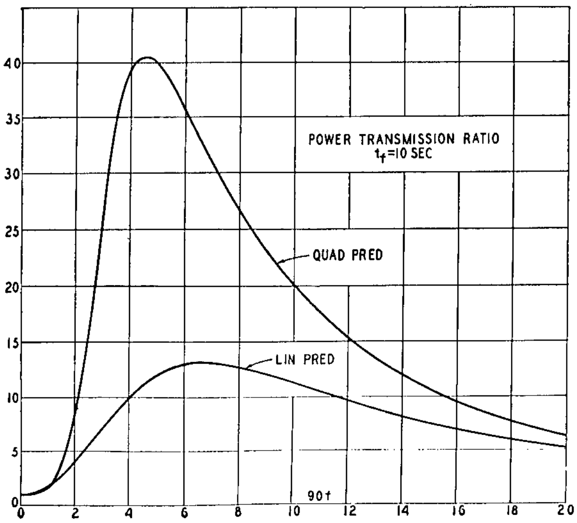


FIGURE 6. Power transmission ratio of linear and quadratic prediction circuits with 10-second prediction time.

TABLE 2

	$t_f = 5$		10		15		20		
$90f$	$\lvert Y_l\rvert^2$	$\lvert Y_q\rvert^2$	$\lvert Y_l\rvert^2$	$\lvert Y_q\rvert^2$	$\lvert Y_l\rvert^2$	$\lvert Y_q\rvert^2$	$\lvert Y_l\rvert^2$	$\lvert Y_q\rvert^2$	P^* Mk-VII
0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	31.4
1	1.29	1.13	1.82	1.60	2.59	2.71	3.59	4.81	33.5
2	2.10	2.76	4.08	8.90	6.97	23.16	10.74	50.35	35.7
3	3.20	6.85	7.19	26.73	12.96	72.51	20.51	159.43	19.7
4	4.2	10.0	10.1	39.5	18.6	106.1	29.76	231.3	3.6
5	5.0	10.5	12.1	39.9	22.4	104.4	35.9	223.9	2.5
6	5.3	9.8	13.1	35.6	24.3	90.6	38.9	190.6	1.2
7	5.4	8.8	13.2	30.8	24.6	76.6	39.4	158.4	1.6
8	5.2	7.9	12.8	26.6	23.8	64.7	38.2	131.8	2.1
9	5.0	7.1	12.2	23.0	22.5	55.0	36.0	110.6	1.4
10	4.7	6.3	11.4	20.0	21.0	47.0	33.5	93.5	0.7
11	4.4	5.7	10.5	17.5	19.3	40.4	30.8	79.6	0.8
12	4.1	5.1	9.7	15.3	17.7	35.0	28.3	68.2	0.8
13	3.8	4.6	8.9	13.5	16.3	30.4	25.8	58.9	0.5
14	3.6	4.2	8.2	12.1	14.9	27.1	23.6	52.0	0.3
15	3.4	3.8	7.6	10.6	13.7	23.4	21.6	44.5	0.8
16	3.2	3.5	7.0	9.5	12.6	20.6	19.8	39.0	1.1
17	3.0	3.2	6.5	8.5	11.6	18.3	18.2	34.4	0.8
18	2.8	3.0	6.0	7.7	10.7	16.3	16.8	30.4	0.4
19	2.7	2.8	5.6	7.0	9.9	14.6	15.5	27.0	0.7
20	2.5	2.6	5.3	6.3	9.2	13.1	14.4	24.1	1.0
rms error of prediction	23.9	29.5	33.9	53.4	44.5	85.4	55.4	125.0	

* P is in units of 180 yd² per c.

Time of flight in seconds	Rms error of prediction due to tracking errors in yards	
	Linear	Quadratic
5	23.9	29.5
10	33.9	53.4
15	44.5	85.4
20	55.4	125.0

It is obviously relatively disadvantageous to use quadratic prediction when the target is in fact flying a rectilinear unaccelerated course.

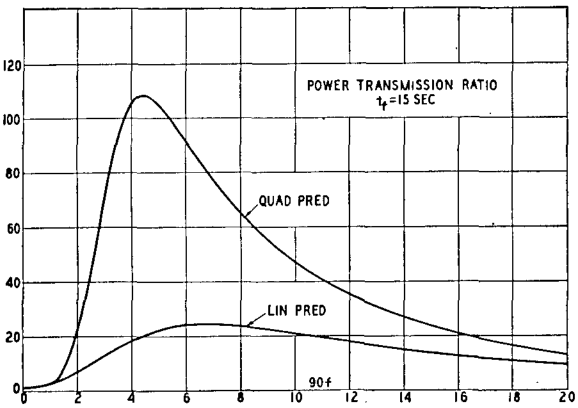


FIGURE 7. Power transmission ratio of linear and quadratic prediction circuits with 15-second prediction time.

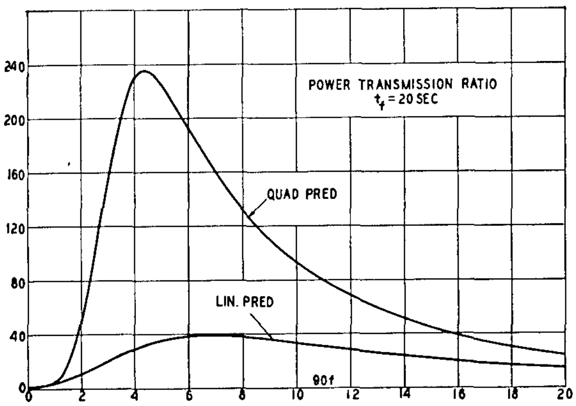


FIGURE 8. Power transmission ratio of linear and quadratic prediction circuits with 20-second prediction time.

The relative advantage of linear prediction should persist for target paths with only a slight amount of curvature, but this relative advantage should decrease as the curvature is increased. When the curvature exceeds a certain amount, the relative advantage should shift to quadratic prediction.

The determination of the minimum value of target path curvature at which quadratic prediction becomes relatively advantageous depends not only upon:

1. dispersion of the predicted point of impact due to tracking errors,

but also upon a number of other factors, among which are:

2. actual future position of target with respect to the predicted point of impact, assuming an accurate computer and the absence of all sources of dispersion enumerated here;[c]

3. dispersion due to inaccuracies in the computer and data-transmission systems;

4. dispersion due to noise in the computer and data-transmission systems;

5. dispersion due to variations in actual dead time;

6. dispersion due to gun wear and to variations in powder charge, shell weight, shell shape, etc.;

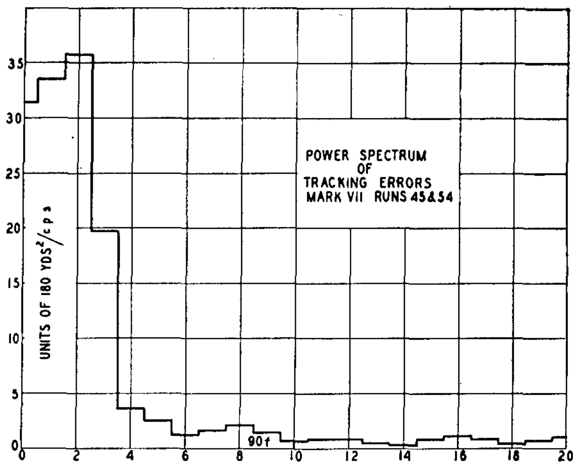


FIGURE 9. Composite power spectrum of tracking errors of experimental radar.

7. dispersion due to variations in meteorological conditions along the path of the shell;

8. dispersion due to variability of time-fuze calibration; and

9. lethal pattern of shell burst.

In a special illustrative case, a numerical analysis, including most of these factors (estimated), showed that quadratic prediction becomes relatively advantageous when the target acceleration exceeds about $0.1g$. However, this should not be taken as a general result.

[c] This is considered in detail in the next section.

13.1.4 Linear and Quadratic Prediction Errors on Constant-Velocity Circular Courses

The use of a finite number of derivatives of the tracking data for purposes of prediction is itself a source of prediction errors even if there were no tracking errors. Definite evaluation of these prediction errors can be made only if the path of the target is prescribed. The simplest path which can be prescribed for this purpose is a circular one at constant velocity. Such a path is fairly realistic when considered in relation to the difficulty of maneuvering a bomber and to actual records of the paths of hostile bombers over London during World War II.

The position of a target flying in a circle at constant velocity, referred to the center of the circle, is expressed by the complex quantity $Re^{i\omega t}$ where R is the radius of the circle and ω is the angular rate. In terms of the velocity V and the transverse acceleration A, we have $R = V^2/A \quad \omega = A/V$. The predicted position is then at $RY(i\omega)e^{i\omega t}$ where $Y(i\omega)$ is the transmission function of the prediction circuit. The true future position of the target, however, is at $R \exp\,[i\omega(t + t_f)]$. Hence, the prediction error, referred to axes fixed on the target and oriented respectively transverse to and in the direction of the present velocity, is

$$\epsilon = R[Y(i\omega) - e^{i\omega t_f}]\,.$$

As an illustration let us consider a case in which $V = 150$ yd per sec, $A = 5$ yd per sec^2 and $t_f = 10$. For the linear prediction circuit

$$Y_l(i\omega) = 1.0409 + i0.3296$$

and for the quadratic prediction circuit

$$Y_q(i\omega) = 0.9501 + i0.3610$$

while

$$e^{i\omega t_f} = 0.9450 + i0.3272\,.$$

Hence, when the present position of the target is at $4500 + i0$ with respect to the center of the circle, the linear predicted point is at $4684 + i1483$, the quadratic predicted point is at $4276 + i1624$ while the true future position is at $4252 + i1472$. These are shown in Figure 10. The prediction error vectors are

$$\epsilon_l = 432 + i11 \qquad |\epsilon_l| = 432$$

$$\epsilon_q = \ \ 24 + i152 \qquad |\epsilon_q| = 154\,.$$

Referring to Figure 10 it may be observed that if the first-derivative component of the prediction were to be reduced by approximately 10 per cent a nearly perfect hit would be obtained. This suggests the possibility of deter-

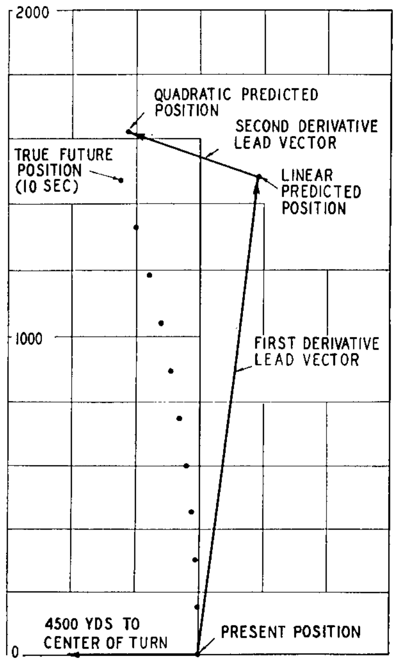


FIGURE 10. Vector diagram of linear and quadratic prediction for constant-velocity circular courses.

mining empirical functions of the time of flight for the potentiometer factors G_1 and G_2 in order to improve the probability of kill. This would involve consideration of all of the sources of dispersion enumerated in the preceding section as well as a statistical study of target paths. Such a determination has not been attempted.

13.1.5 Physical Configuration of the Second-Derivative Circuit

In this section we shall derive a physical configuration for the second-derivative circuit. In particular it illustrates the application of feedback to the realization of weighting functions or impulsive admittances involving complex exponentials in general.[d] It should be pointed out, however, that the application of feedback to the end in view is not restricted to purely

[d] Originally proposed by R. L. Dietzold.

electronic circuits. An application involving the use of servomechanisms will be described in Section 13.4.

The transmission function which concerns us here may be expressed in the partially factored form

$$Y(p) = \frac{p^2}{(p + 0.2087)(p + 1.5864)(p^2 + 0.3049p + 0.0666)}$$

where the poles have been adjusted to correspond to $T = 20$ seconds and where a constant factor has been left out.

The circuit is to be designed to work out of the amplifier in the first-derivative circuit of the M9 director. Since this much of the first-derivative circuit has a transmission function of the form $p/(p+0.24)$, the transmission function which we have to realize is $Y_i(p)/Y_f(p)$ where

$$Y_i(p) = \frac{p}{(p + 0.2087)\ (p + 1.5864)}$$

and

$$Y_f(p) = \frac{p^2 + 0.3049p + 0.0666}{p + 0.24}\ .$$

The inversion of the factor corresponding to $Y_f(p)$ is in accordance with the fact that the transmission gain through a feedback amplifier is equal to the loss in the feedback network, provided the feedback is very large. To realize the transmission function $Y_i(p)/Y_f(p)$ it is therefore necessary only to realize the transmission functions $Y_i(p)$ and $Y_f(p)$ individually. The corresponding networks are shown in Figure 11, with typical element values.

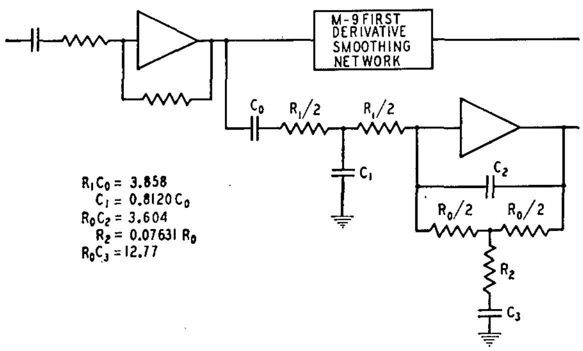


FIGURE 11. Physical configuration of quadratic prediction circuit for modified M9 AA director.

The input network has four elements, whereas $Y_i(p)$ has only two parameters. Hence there are two degrees of freedom in the element values of this network. One degree of freedom must be reserved for the impedance level; the other permits some latitude in the relative values of the resistances and stiffnesses.

The feedback network has four independent elements, whereas $Y_f(p)$ has three parameters. Hence there is only one degree of freedom in the element values of this network. This degree of freedom must be reserved for the impedance level.

There is, however, one degree of freedom between the impedance levels of the two networks. This follows from the fact that the transmission function of the circuit is the ratio of the transmission functions of the individual networks. The scale factor for the transmission function of the circuit is readily determined from the fact that the transmission function must be approximately pR_0C_0 at small values of p.

13.2 CIRCUIT FOR CLOSE SUPPORT PLOTTING BOARD

In this application, position data smoothing with delay correction for constant rates of change in position was required. Assuming flat random noise in position data, and, arbitrarily, 1-second smoothing time, the best transmission function for position data smoothing without delay correction is $y_0(p)$ in the notation of Section 11.3. The best transmission function for the first-derivative circuit, if it were required, is $py_1(p)$. Hence, the best transmission function for position data smoothing with full delay correction is

$$Y_I(p) = y_0(p) + \frac{1}{2}\,py_1(p)\ .$$

This corresponds to the weighting function

$$\begin{aligned} W_I(t) &= w_0(t) + \frac{1}{2}\,\dot{w}_1(t) \\ &= 2(2 - 3t) \quad 0 < t < 1\ . \end{aligned}$$

The series expansion for $Y_I(p)$ is, by (15) of Chapter 11,

$$Y_I(p) = 1 - \frac{p^2}{12} + \frac{p^3}{30} - \frac{p^4}{120} + \cdots\ .$$

The form of the rational approximation was chosen as

$$Y(p) = \frac{1 + a_1 p}{1 + b_1 p + b_2 p^2 + b_3 p^3}$$

in order to obtain a loss characteristic which has an ultimate slope of 12 db per octave.[e] This requirement was also set as a precaution against noise due to granularity of the coordinate-conversion potentiometers. The coefficients are determined by

$$b_1 = a_1$$

$$b_2 - \frac{1}{12} = 0$$

$$b_3 - \frac{1}{12}b_1 + \frac{1}{30} = 0$$

$$-\frac{1}{12}b_2 + \frac{1}{30}b_1 - \frac{1}{120} = 0$$

whence

$$Y(p) = \frac{1 + \frac{11}{24}p}{1 + \frac{11}{24}p + \frac{1}{12}p^2 + \frac{7}{1440}p^3}.$$

This may be expressed in the form $Y(p) = Y_i(p)/Y_f(p)$ where

$$Y_i(p) = \frac{1}{1 + 0.1053p}$$

$$Y_f(p) = \frac{1 + 0.3530p + 0.04615p^2}{1 + 0.4583p}.$$

The circuit configuration is shown below in Figure 12.

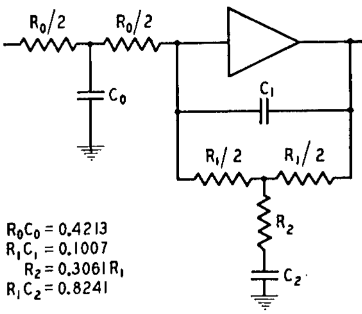


FIGURE 12. Physical configuration of data-smoothing circuit for close support plotting board.

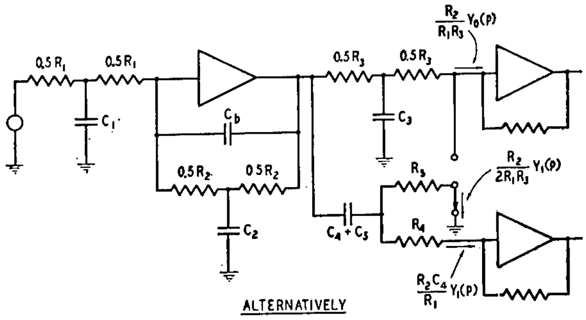
[e] This design also antedated the formulation of the $n - m = r + 1$ rule given in Section 12.2 according to which we should have taken $Y_I(p) = y_1(p) + \frac{1}{2}\, p y_2(p)$.

13.3 CIRCUIT FOR GROUND-CONTROL BOMBING COMPUTER

In this application, rate smoothing as well as position smoothing was required. In addition, delay correction in position, for constant rate of change, was to be available but optional, and the loss characteristic was to have an ultimate slope of 12 db per octave, or more.

In accordance with the $n - m = r + 1$ rule, the best transmission function for position data is $y_1(p)$, whereas that for rate is $py_2(p)$. A number of designs were made on this basis. However, from the point of view of network economy they were inferior to a design based on $y_2(p)$ for position data. The use of $y_2(p)$ for position data is not consistent, theoretically, with the use of $py_2(p)$ for rate, but the practical advantage outweighs the theoretical disadvantage.

The rational approximation used for $y_2(p)$



	ALTERNATIVELY	
R_1C_1 = 0.8613	0.5438	
R_2C_b = 0.2847		
R_2C_2 = 0.5438		
R_3C_3 = 0.5438	0.8613	
R_3C_5 = 0.5000		(FOR DELAY CORRECTION)
R_4C_4 = 0.1360	0.2153	(FOR FIRST DERIVATIVE)
R_5C_5 = 0.1360	0.2153	(FOR DELAY CORRECTION)

FIGURE 13. Physical configuration of linear prediction circuit for ground-control bombing computer.

is the one given in (6), Section 12.2. It may be expressed as

$$y_2(p) = \frac{Y_{i1}(p) \cdot Y_{i2}(p)}{Y_f(p)}$$

where

$$Y_{i1}(p) = \frac{1}{1 + 0.2153p}$$

$$Y_f(p) = \frac{1 + 0.2847p + 0.03870p^2}{1 + 0.1359p}$$

$$Y_{i2}(p) = \frac{1}{1 + 0.1359p}.$$

It may be noted that a redundant factor has been introduced, viz., $1 + 0.1359p$, in order to secure a physically realizable $Y_f(p)$. The coefficient was chosen so that a resistance would not be required in the shunt branch of the feedback network. Referring to the circuit configuration in Figure 13, the transmission function of the input network is $Y_{i1}(p)$, that of the feedback network is $Y_f(p)$, and that of the output network at the top is $Y_{i2}(p)$.

The output impedance of the amplifier is reduced nearly to zero by virtue of shunt feedback.[15j] Hence, the rate circuit, as shown in Figure 13, may be derived from the amplifier output through a simple additional network whose transmission function is $pY_{i2}(p)$. Two rate outputs are provided so that the delay introduced in position may be corrected optionally without disturbing scale factors.

13.4 CIRCUIT USING SERVOMECHANISMS

In the final report, October 25, 1945, to NDRC Division 7, on the research program carried on under Contract NDCrc-178, a list is given of a number of the more important practical advantages for the use of a-c carrier in computing circuits. These advantages are:

1. Permits operation at lower levels before running into trouble with thermal noise, contact potentials, drifts due to temperature;
2. Permits use of transformers for impedance matching, voltage transformations, coupling between balanced and unbalanced circuits;
3. Permits use of hybrid coils for voltage summations of moderate precision;
4. Eliminates the necessity for modulators in servo circuits using a-c motors;
5. Permits reduction in total power consumption, rectified power for amplifiers, and voltage regulation.

However, the techniques of differentiation and of data smoothing with fixed networks in computing circuits which use d-c carrier, are not applicable to computing circuits which use a-c carrier.

The circuit described here is an example of one of the techniques used in the T15-E1 experimental curved flight director.[6] In Figure 14 servo motors[f] are indicated by M, and generators by G. The motors are two-phase induction motors with one phase winding of each energized directly by the carrier source at constant amplitude. The generators are essentially two-phase induction motors also with one phase winding of each energized directly by the carrier source at constant amplitude. They deliver, at

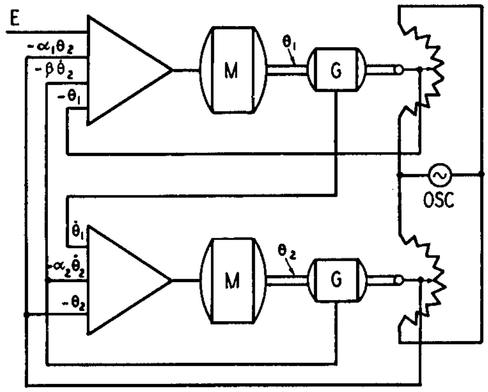


FIGURE 14. Electromechanical linear prediction circuit.

the other phase windings, carrier voltage at amplitudes proportional to the angular velocities $\dot{\theta}_1$ and $\dot{\theta}_2$ of the shafts. The potentiometers are energized by the carrier source at constant amplitude. They deliver carrier voltage at amplitudes proportional to the angular positions θ_1 and θ_2 of the shafts from some reference positions. The position data are represented by the modulation amplitude E.

With amplifiers of sufficiently large voltage gain and power capacity, and motors of sufficiently large torque, the operational equations of the circuit are readily found by equating to zero the sum of the voltages applied to each amplifier. Thus

$$\theta_1 + (\alpha_1 + \beta p)\theta_2 = E$$
$$p\theta_1 - (1 + \alpha_2 p)\theta_2 = 0$$

whence

$$\theta_1 = \frac{1 + \alpha_2 p}{1 + (\alpha_1 + \alpha_2)p + \beta p^2} E$$

$$\theta_2 = \frac{p}{1 + (\alpha_1 + \alpha_2)p + \beta p^2} E .$$

The angular position θ_1 therefore represents the smoothed position data while the angular position θ_2 represents the smoothed rate.

[f] The technique of using servo motors for smoothing, as described above, is due chiefly to E. L. Norton.

Chapter 14

VARIABLE AND NONLINEAR CIRCUITS

THE PAST DISCUSSION has been more or less clearly directed at predictor systems having certain well-defined properties. For example, it has been tacitly assumed that the first part of the prediction system will consist of geometrical manipulations transforming the raw input data into other quantities, such as the components of velocity in Cartesian or intrinsic coordinates, which we have some physical reason to believe should be approximately constant for extended periods.[a] These quantities, then, are isolated explicitly in the circuit and are the actual effective inputs of the data-smoothing networks. The data-smoothing networks themselves are, of course, definitely assumed to be linear and invariable.

This is obviously a straightforward attack but it does not necessarily exhaust all possibilities. For example, advantages may be gained by using data-smoothing networks which are nonlinear or which vary with time or target position. It may also be possible to smooth the input data according to some geometric assumption, such as straight line flight, without the necessity of isolating geometrical parameters explicitly.

This chapter attempts to illustrate these possibilities by some rather scattered examples. Data-smoothing networks which vary with time seem to give improved performance over fixed networks, and have been studied with some care. Several examples are given at the end of the chapter. None of the other lines, however, has been explored at all thoroughly. The examples of data-smoothing networks variable with time are, in a sense, illustrations of nonlinearity also, since they all operate on the assumption that the cycle of the network's variation with time begins anew at each marked change in course. Since a change in course is exactly like a tracking error, except that it is much larger, this resetting requires a nonlinear control circuit which will respond to large amplitude effects but not to small ones. This, however, is evidently a very mild sort of nonlinearity. More thoroughgoing nonlinearities have not been studied. There seems to be no *a priori* reason for supposing that they would appreciably improve the performance of data-smoothing networks.

The first part of the chapter gives examples of data-smoothing schemes which do not require the isolation of geometrical parameters. They are based on degenerative feedback circuits which satisfy the requisite formal relations but which might, in some cases, be unstable in practice. This portion of the material is included primarily for its possible suggestive value rather than for its concrete practical usefulness.

14.1 THE PROTOTYPE FEEDBACK CIRCUIT

The diversity of particular circuits can be given a certain unity by regarding them all as modifications of the feedback smoothing circuit shown originally in Figure 2 of Chapter 10. In accordance with the discussion of that figure it will be convenient to suppose that the resistive feedback path is introduced to limit the gain of the amplifier proper, so that the structure reduces to an amplifier with high but finite gain and a pure capacity feedback. The circuit has a net loop gain, and is consequently degenerative, at any moderately high frequency. For our present purposes, it is convenient to recall the general property of degenerative feedback amplifiers, that they tend to suppress any given frequency by the amount of the degenerative feedback for that frequency. This suppression obtains not only at the amplifier output but at many other points in the circuit as well. For example, it holds at the amplifier input if we combine the original applied voltage with the voltage contributed by the feedback[b] circuit.[15g] Thus, except for the absolute

[a] This is true ideally even in the Wiener system since Wiener assumes that transformations will be made to some suitable coordinate system, preferably the intrinsic, before the statistical prediction method is applied.

[b] This follows immediately from the fact that, since the characteristics of the amplifier proper are not changed by the addition of the feedback path, the output voltage is always a fixed multiple of the net input voltage.

signal level, it is not necessary to transmit through the amplifier of Figure 2 of Chapter 10 in order to produce the smoothing effect. It would be sufficient to hang the input circuit of the amplifier, as a two-terminal impedance, across the circuit.

14.2 SIMULTANEOUS SMOOTHING IN THREE COORDINATES

The property of degenerative feedback circuits which has just been described is conveniently illustrated by a three-dimensional extension of the original smoothing circuit of Figure 2 of Chapter 10. The three-dimensional circuit is shown in Figure 1. The three input voltages are the quantities D, $D\dot{E}$, and $D\dot{A}$ cos

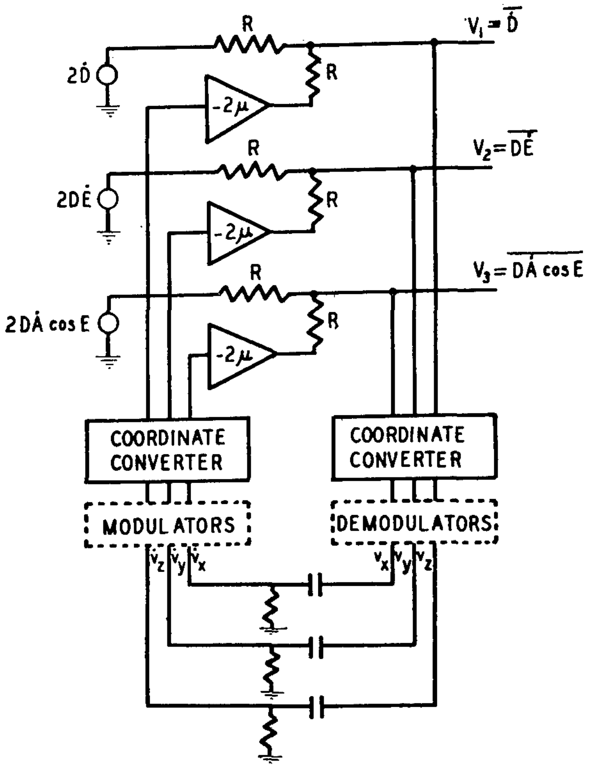


FIGURE 1. Feedback smoothing in three coordinates.

E, where D, E, and A are, respectively, slant range, elevation, and azimuth. The three voltages will be recognized as the three components of the target motion in a tilted and rotating rectangular coordinate system. One axis of the tilted system is directed along the instantaneous line of sight to the target and the other two are perpendicular to this one in the vertical and horizontal planes respectively.[c] It is assumed that these input rates represent target motion in a straight line, plus the usual tracking errors. The object of the smoothing system is to provide shunt impedances which will tend to suppress the tracking errors by feedback action, according to the principles described in the preceding section, without disturbing the portions of the input voltages corresponding to the assumed straight line path.

We can simplify the analysis by restricting our attention to the special case of two-dimensional motion which occurs when the target course lies in a vertical plane passing directly through the antiaircraft position. This is illustrated in Figure 2. In this case the component $DA \cos E$ is evidently zero. If we represent the voltage at the other two terminals, including both the original applied voltages and the voltages fed back through the circuit, by V_1 and V_2, the voltages coming out of the coordinate converter on the right-hand side in Figure 2 are

$$\begin{aligned} v_x &= V_1 \cos E - V_2 \sin E \\ v_y &= V_2 \cos E + V_1 \sin E \,. \end{aligned} \tag{1}$$

These voltages are differentiated, passed through a second coordinate converter, and fed back so that the output voltages must satisfy the equations

$$\begin{aligned} V_1 &= \dot{D} - \mu(\dot{v}_x \cos E + \dot{v}_y \sin E) \\ V_2 &= D\dot{E} - \mu(\dot{v}_y \cos E - \dot{v}_x \sin E) \,. \end{aligned} \tag{2}$$

In order to exhibit the smoothing action of the circuit let us denote the observed velocity components, referred to the upright and fixed

[c] This is the coordinate system which was used in the experimental T15 director. A complete prediction circuit can be obtained by using the three voltages described here as inputs to the lead servos in the T15 system. In the actual T15 system, rates in the tilted and rotating coordinate system were obtained by the so-called "memory point" method. The voltages $\dot{D}$, $D\dot{E}$, etc., required with the present method, might be obtained with the help of tachometers attached to the tracking shafts to measure the instantaneous values of $\dot{D}$, $\dot{E}$, and $\dot{A}$. An equivalent to the variable smoothing of the memory point method can be obtained by making the gains in the feedback paths in Figure 1 variable according to the principles described in a later section.

rectangular coordinate system, by u_x and u_y, so that

$$u_x = \dot{D} \cos E - D\dot{E} \sin E$$

$$u_y = D\dot{E} \cos E + \dot{D} \sin E \,. \tag{3}$$

Substituting (2) and (3) into (1), we get

$$v_x = u_x - \mu\dot{v}_x$$

$$v_y = u_y - \mu\dot{v}_y$$

or

$$\mu\dot{v}_x + v_x = u_x$$

$$\mu\dot{v}_y + v_y = u_y \,.$$

These show clearly that v_x and v_y are smoothed values of u_x and u_y, respectively. If μ is constant the smoothing is of fixed exponential type. If μ is proportional to the time up to some maximum value, the smoothing is of the variable type described in Sections 14.6 and 14.7.

To complete the discussion of the circuit we observe that by (1)

$$V_1 = v_x \cos E + v_y \sin E$$

$$V_2 = v_y \cos E - v_x \sin E \,.$$

These show that V_1 and V_2 are the smoothed rate components referred to the tilted and rotating rectangular coordinate system. The fact that the orientation of this coordinate system, which depends upon the observed angular height E, is not smoothed makes no difference to the computation of the leads because this computation is made instantaneously in the same coordinate system to which the smoothed rate components are instantaneously referred.

The analysis in the general case including all three coordinates is of the same nature. Since the rate components in fixed rectangular coordinates appear in the middle of the feedback path, it is perhaps not fair to regard the circuit as an illustration of a data-smoothing device which does not rely upon the explicit isolation of the geometrical parameters of the assumed target path. It should be pointed out, however, that in comparison with a straightforward geometrical solution in which velocity components in fixed coordinates are first isolated explicity, then smoothed, and then used to form the basis of prediction, the circuit in Figure 1 has the advantage that most of the components can be built with very low precision. What is transmitted around the feedback loop is essentially the tracking errors only. Since tracking errors are always small, very high percentage errors in the system can be tolerated.[d]

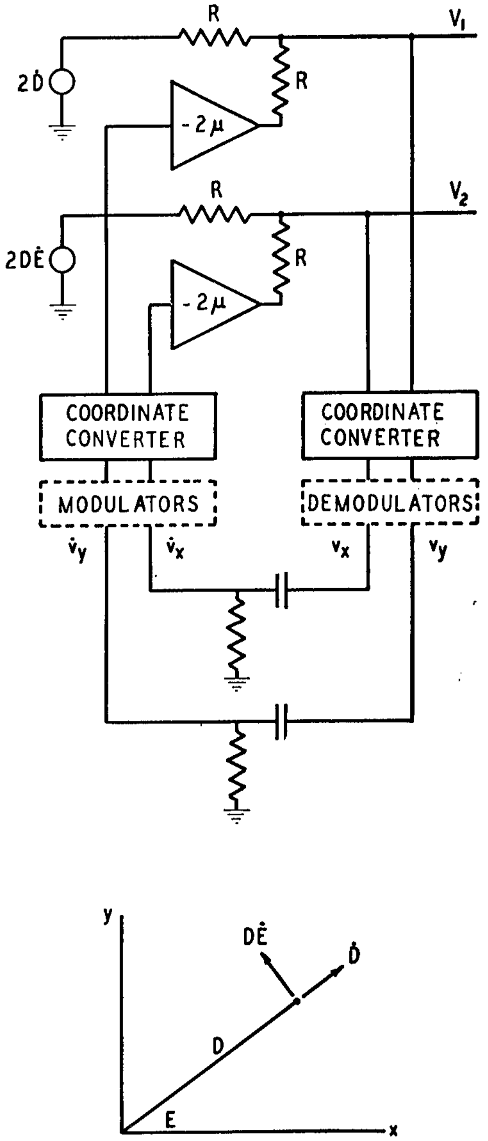


FIGURE 2. Feedback smoothing in two coordinates.

14.3 SMOOTHING NETWORKS VARIABLE WITH TARGET POSITION

It was mentioned earlier that changing the data-smoothing network with the target coordinates represented one way in which the results obtained from fixed networks could be

[d] An exception to this statement must be made for errors in the coordinate converters which fluctuate rapidly with target position.

generalized. In a sense, the coordinate conversions of Figure 1 are illustrations of these possibilities. A better illustration, however, is provided by the circuit of Figure 3. The structure is intended to give smooth slant range rate from slant range data, under the assumption of unaccelerated straight line target motion.

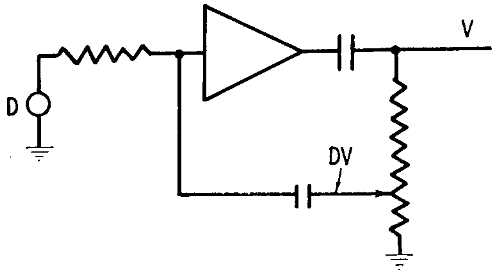


FIGURE 3. Feedback smoothing with smoothing variable with position coordinates.

The relation between input and output in Figure 3 is readily seen to be[e]

$$V = \frac{d}{dt}\left[D - \mu \frac{d}{dt}(DV)\right]$$

or

$$\mu \frac{d^2}{dt^2}(DV) + V = \frac{dD}{dt} \qquad (4)$$

where μ is the amplifier gain, D is slant range, and $V = dD/dt$ is slant range rate.

The principle of the circuit depends upon the fact that under the assumed target motion the square of the slant range, D^2, should be a quadratic function of time, so that $[D(dD/dt)]$ should be a linear function of time and $(d/dt)[D(dD/dt)]$ should be a constant. This last is the quantity which is fed back in Figure 3. If it actually is a constant, it has no further influence on the calculation, since the forward circuit includes a differentiator, and the operation of the circuit is the same as though no feedback term were present. This can be verified by setting $D = D_0 = \sqrt{a + 2bt + ct^2}$, corresponding to ideal straight line flight, in equation (4). It is readily seen that the equation is satisfied by

$$V = V_0 = \frac{b + ct}{\sqrt{a + 2bt + ct^2}} = \frac{dD_0}{dt},$$

the first or feedback term being zero.

[e] The condensers in Figure 3 symbolize differentiation.

If D does not correspond exactly to straight line flight, either because of tracking errors or actual target maneuvers, on the other hand, the feedback voltage is no longer constant. In this case transmission around the loop can exist and the degenerative feedback action produces smoothing in both the input and the output voltage. In calculating the exact effect we must take account of the fact that the feedback voltage depends upon the D potentiometer in the feedback circuit as well as upon the output voltage V. Since the D potentiometer setting must include the errors in the input data, this means that the output voltage is not perfectly smoothed, even with unlimited gain around the loop. The percentage error in the output rate tends in the limit to approximate the percentage error in D itself. For practical purposes, however, this is a very satisfactory result, since in the absence of smoothing percentage errors in rates are usually many times those of the corresponding coordinates.

It is apparent that it should be possible to construct many circuits of this general type from the differential equations of the trajectory. A second example is furnished by Figure 4. The operation of the circuit is essentially

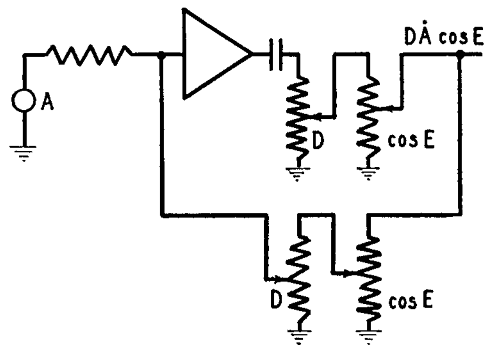


FIGURE 4. Another example of feedback smoothing with smoothing variable with position coordinates.

similar to that of Figure 3. It depends upon the fact that in unaccelerated straight line motion the quantity $D^2\dot{A}\cos^2 E$ is a constant. Instead of multiplying by D^2 and $\cos^2 E$ at a single point in the feedback loop, however, separate multiplications by D and $\cos E$ are introduced in the forward and feedback circuits. This permits the output to appear as a smoothed value of the quantity $D\dot{A}\cos E$, which will be recalled as one of the primary quantities in the circuit of Figure 1.

14.4 NETWORKS VARIABLE WITH TIME

In addition to making the parameters of the data-smoothing network vary as functions of the coordinates of target position we may also make them variable as functions of time. The advantage of variation with time can be understood by going back to the discussion of the analytic arc assumption and its consequences for fixed data-smoothing networks, as given in Chapters 9, 10, and 11. It will be recalled that for any given settling time there was an optimum choice of the network's weighting function. The choice of the settling time itself, however, was always a compromise. On the one hand, making the settling time too short led to too little smoothing, so that the dispersion in the resulting fire became excessive. On the other hand, too long a settling time meant that data from previous unrelated segments were retained in the smoothing circuit during too large a proportion of an average individual segment of the target path, leaving too small a residue of the average segment as useful firing time.

It is evident that it is theoretically possible to escape the consequences of this compromise by resorting to variable structures. We need merely assume that the network always has a weighting function appropriate for a settling time equal to the time since the last change in course. This would give a small amount of smoothing shortly after a change in course, with more smoothing and consequently greater accuracy later on. No firing time, however, is sacrificed waiting for the network to settle.

In order to exploit these possibilities we must, of course, be able to design networks to give at least approximately the right sequence of weighting function. It is also necessary to provide some sort of auxiliary controlling mechanism which will sense changes in target course and return the variable circuits in the smoothing network proper to their initial positions. These are both difficult problems which have been incompletely explored. Some elementary solutions, based principally upon modifications of the degenerative feedback smoothing circuit of Figure 2, of Chapter 10, are, however, given later in the chapter. As a preliminary, the next section gives a formal extension of the general polynomial expansion method of Chapter 11 to the variable case.

14.5 GENERAL POLYNOMIAL SOLUTION FOR VARIABLE NETWORKS

The extension of the general method of Chapter 11 to the variable case requires two modifications.

1. The lower limit of the integral to be minimized is now taken as zero, in anticipation of the possibility of discriminating between relevant and irrelevant data on the basis of time of arrival.

2. The weighting function may now depend more generally upon the variable of integration and the upper limit of integration.

With these modifications there is no longer any advantage in conducting the analysis in terms of the age variable τ. To deal directly with the minimization of the integral

$$\int_0^t [\bar{E}(\lambda) - E(\lambda)]^2 W_0(t,\lambda)\, d\lambda\ , \tag{5}$$

let

$$\bar{E}(\lambda) = V_0 + V_1 \cdot G_1(t,\lambda) + \cdots + V_n \cdot G_n(t,\lambda), \tag{6}$$

Where $G_m(t,\lambda)$ is an mth degree polynomial in λ. Also, let

$$\int_0^t W_0(t,\lambda)\, d\lambda = 1$$

$$\begin{aligned}\int_0^t G_l(t,\lambda) \cdot G_m(t,\lambda) \cdot W_0(t,\lambda)\, d\lambda &= 0 \qquad \text{if } l \neq m \\ &= \frac{1}{k_m} \qquad \text{if } l = m\end{aligned}$$

$$(G_0 = 1,\ k_0 = 1)\ .$$

Then (5) is a minimum with respect to the V_m's in (6) if

$$V_m(t) = \int_0^t E(\lambda) \cdot W_m(t,\lambda)\, d\lambda \tag{7}$$

where

$$W_m(t,\lambda) = k_m G_m(t,\lambda) \cdot W_0(t,\lambda)\ . \tag{8}$$

The possibility of physically realizing the $V_m(t)$ depends upon the possibility of realizing networks with impulsive admittances $W_m(t,\lambda)$ in the sense that $W_m(t,\lambda)$ is the response of a network, at time t, to a unit impulse applied at time λ, where $0 \le \lambda \le t$. Taking this possibility for granted, the predicted value $\bar{E}(t + t_f)$ is, according to (6), a variable linear combination of the $V_m(t)$, viz.,

$$\bar{E}(t + t_f) = V_0(t) + G_1(t,t + t_f) \cdot V_1(t) + \cdots + G_n(t,t + t_f) \cdot V_n(t). \quad (9)$$

It is clear that all of the $W_m(t,\lambda)$ as well as all of the $G_m(t,\lambda)$ for $m = 1, 2, \cdots$ are determined by $W_0(t,\lambda)$. The latter is determined as the best weighting function for position data smoothing, depending upon the characteristics of the noise associated with the position data. The general methods of determining the best weighting function with fixed smoothing time, described in Chapter 10, may be used to determine the best weighting function with variable smoothing time.

Under the assumption that the spectrum of the noise associated with the signal $S(t)$ has a uniform slope of $6k$ db per octave, we may take over from Section 11.3 the result that the best weighting function is

$$w_k(t,\lambda) = \frac{(2k + 1)!}{(k!)^2 t} \left[\frac{\lambda}{t}\left(1 - \frac{\lambda}{t}\right)\right]^k \quad (10)$$

$$0 \le \lambda \le t .$$

The response of the network is then

$$V(t) = \int_0^t S(\lambda) \cdot w_k(t,\lambda) \, d\lambda . \quad (11)$$

14.6 SPECIAL CASES

It will be illuminating to consider a few special cases of (11).

For $k = 0$, we have

$$V(t) = \frac{1}{t} \int_0^t S(\lambda) \, d\lambda . \quad (12)$$

Multiplying through by t and differentiating we get

$$t\dot{V}(t) + V(t) = S(t) . \quad (13)$$

This suggests the circuit shown in Figure 5.[f]

For $k = 1$, we have

$$V(t) = \frac{6}{t^3} \int_0^t S(\lambda) \cdot \lambda(t - \lambda) \, d\lambda .$$

Multiplying through by t^3 and differentiating twice we get

$$\frac{1}{6} t^2\ddot{V} + t\dot{V} + V = S$$

which may be written in the form

$$\left(\frac{t}{2}\frac{d}{dt} + 1\right)\left(\frac{t}{3}\frac{d}{dt} + 1\right) \cdot V = S .$$

This suggests the network shown in Figure 6.[g]

14.7 NETWORKS WITH A LIMITED RANGE OF VARIATION

By generalizing the above results in various ways a large number of other examples of variable smoothing networks can be constructed. Since unlimited variation in the smoothing time is not practically possible, or perhaps even tactically optimal, however, it is desirable in discussing any further examples to include also the possibility that the range of variation in the network may be restricted. For any positive integral value of k in (11) the differential equation for $V(t)$ is of the type which may be reduced by the transformation $t = e^z$ to a linear differential equation with constant coefficients.[h] In general, this facilitates the determination of what happens to the weighting function $w_k(t,\lambda)$ when $t > T$ if the variability of the network is stopped at time T. In the case of the first-order equation (13), however, it is just as easy to deal directly in terms of the natural time.

A more general form for (13), which readily yields the effects of a sudden or gradual stoppage of the variability of the network, is

$$\frac{\phi(t)}{\dot{\phi}(t)} \dot{V}(t) + V(t) = S(t) . \quad (14)$$

This corresponds to the response

$$V(t) = \frac{1}{\phi(t)} \int_0^t S(\lambda) \cdot \dot{\phi}(\lambda) \, d\lambda$$

whence the weighting function is

$$w(t,\lambda) = \frac{\dot{\phi}(\lambda)}{\phi(t)} . \quad (15)$$

[f] This circuit is due to S. Darlington.

[g] Due to B. T. Weber.

[h] See Section A.11 for a more general transformation.

The general relation (14) may be realized with the network of Figure 5, by varying the resistance in accordance with

$$R = \frac{1}{C}\frac{\phi(t)}{\dot{\phi}(t)} \qquad t > 0\,.$$

However, a more practical circuit results from the introduction of variable potentiometers[i] in both the capacity and resistance paths of the original feedback smoothing circuit of Figure 2, Chapter 10. This is shown in Figure 7.[j] It may be noted that the feedback circuit is also applicable to the two cases discussed in the preceding section. It has the advantage for these applications that it does not require the zero-impedance generators and infinite-impedance loads of Figures 5 and 6.

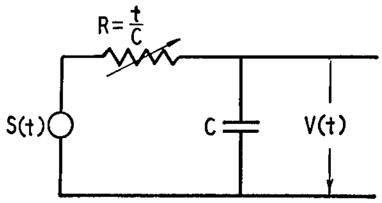


FIGURE 5. Time-variable smoothing circuit giving uniform weighting function.

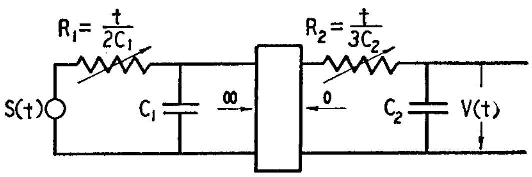


FIGURE 6. Time-variable smoothing circuit giving parabolic weighting function.

As an example of (14) we may take

$$\begin{aligned}\phi(t) &= t \quad 0 < t < T\\ &= Te^{(t-T)/T} \quad t > T\,.\end{aligned}$$

Then

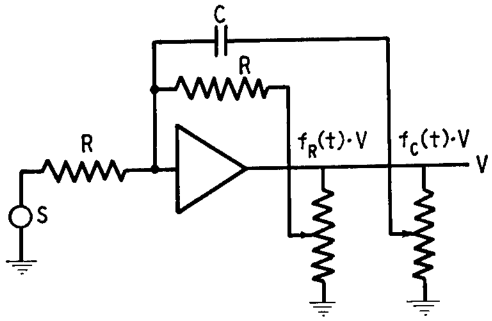
$$\begin{aligned}\frac{\phi(t)}{\dot{\phi}(t)} &= t \quad 0 < t < T\\ &= T \quad t > T\,.\end{aligned}$$

Hence, in Figure 7, if $RC = T$

$$\begin{aligned}f_C(t) &= \frac{t}{T} \quad f_R(t) = 0 \quad 0 < t < T\\ &= 1 \qquad\quad\;\; = 1 \quad t > T\,.\end{aligned}$$

This example obviously calls for a linear potentiometer in the condenser path and a switch in the resistance path. The weighting function obtained is, by (15),

$$\begin{aligned}w(t,\lambda) &= \frac{1}{t} \quad 0 < \lambda < t < T\\ &= \frac{1}{T}\,e^{-(t-T)/T} \quad 0 < \lambda < T < t\\ &= \frac{1}{T}\,e^{-(t-\lambda)/T} \quad 0 < T < \lambda < t\end{aligned}$$



$$f_C(t) = \frac{1}{RC}\frac{\phi(t)}{\dot{\phi}(t)} \qquad f_R(t) = 1 - \frac{d}{dt}\left[\frac{\phi(t)}{\dot{\phi}(t)}\right]$$

FIGURE 7. Limited range time-variable feedback smoothing circuit.

This is illustrated in Figure 8 for $T = 10$, $t = 5$, 10, 20.

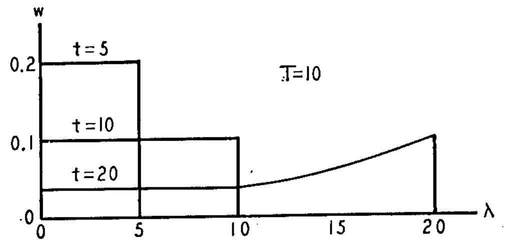


FIGURE 8. First example of weighting function produced by circuit of Figure 7.

A second example is furnished by taking

$$\begin{aligned}\phi(t) &= t^k \quad 0 < t < T\\ &= T^k e^{k(t-T)/T} \quad t > T\,.\end{aligned}$$

Then

$$\begin{aligned}\frac{\phi(t)}{\dot{\phi}(t)} &= \frac{t}{k} \quad 0 < t < T\\ &= \frac{T}{k} \quad t > T\,.\end{aligned}$$

[i] In some cases a variable potentiometer may turn out to be a switch.
[j] This circuit is due to S. Darlington.

Hence in Figure 7, if $RC = T/k$,

$$f_C(t) = \frac{t}{T} \quad f_R(t) = 1 - \frac{1}{k} \quad 0 < t < T$$
$$= 1 \qquad = 1 \quad t > T\,.$$

The first example is a special case of this one. The weighting function obtained is, by (15),

$$w(t,\lambda) = \frac{k\lambda^{k-1}}{t^k} \quad 0 < \lambda < t < T$$
$$= \frac{k\lambda^{k-1}}{T^k} e^{-k(t-T)/T} \quad 0 < \lambda < T < t$$
$$= \frac{k}{T} e^{-k(t-\lambda)/T} \quad 0 < T < \lambda < t\,.$$

This is illustrated in Figure 9 for $k = 3/2$, $T = 10$, $t = 5, 10, 20$.

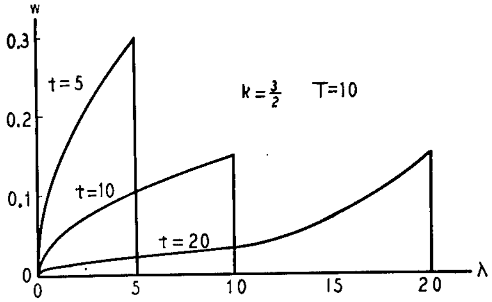


FIGURE 9. Second example of weighting function produced by circuit of Figure 7.

A third example is furnished by taking

$$\phi(t) = \frac{t}{2 - \frac{t}{T}} \quad 0 < t < T$$
$$= Te^{2(t-T)/T} \quad t > T\,.$$

Then

$$\frac{\phi(t)}{\dot\phi(t)} = t\left(1 - \frac{t}{2T}\right) \quad 0 < t < T$$
$$= \frac{T}{2} \quad t > T\,.$$

Hence, in Figure 7, if $RC = T/2$,

$$f_C(t) = \frac{2t}{T}\left(1 - \frac{t}{2T}\right) \quad f_R(t) = \frac{t}{T} \quad 0 < t < T$$
$$= 1 \qquad = 1 \quad t > T\,.$$

The weighting function obtained is, by (15),

$$w(t,\lambda) = \frac{1 - \frac{t}{2T}}{t\left(1 - \frac{\lambda}{2T}\right)^2} \quad 0 < \lambda < t < T$$
$$= \frac{1}{2T\left(1 - \frac{\lambda}{2T}\right)^2} e^{-2(t-T)/T} \quad 0 < \lambda < T < t$$
$$= \frac{2}{T} e^{-2(t-\lambda)/T} \quad 0 < T < \lambda < t\,.$$

This is illustrated in Figure 10 for $T = 10$, $t = 5, 10, 20$.

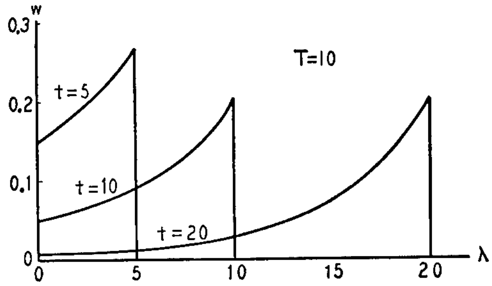


FIGURE 10. Third example of weighting function produced by circuit of Figure 7.

A fourth example is furnished by taking

$$\phi(t) = e^{kt} - 1 \quad t > 0\,.$$

Then

$$\frac{\phi(t)}{\dot\phi(t)} = \frac{1}{k}(1 - e^{-kt}) \quad t > 0\,.$$

Hence, in Figure 7, if $RC = 1/k$,

$$f_C(t) = f_R(t) = 1 - e^{-kt} \quad t > 0\,.$$

The weighting function obtained is, by (15),

$$w(t,\lambda) = \frac{k}{1 - e^{-kt}} e^{-k(t-\lambda)} \quad 0 < \lambda < t\,.$$

For any value of t this weighting function is exponential in λ.

14.8 OTHER EXAMPLES

Because there has been no demand for variable networks in the field of communications, the technique of designing practical variable networks is in a very rudimentary stage compared to that of designing fixed networks. In the remainder of this chapter we shall describe some of the circuits which have been developed for specific practical applications.

A memory point method of obtaining smoothed rates, based upon (12), is illustrated below. If $S(t)$, the quantity to be smoothed, represents the time derivative $\dot{E}(t)$ of the position data $E(t)$, then the average rate is given by

$$V(t) = \frac{E(t) - E(0)}{t} . \tag{16}$$

Under the assumption that the position data, aside from tracking errors, is a linear function of time, the average rate is also the smoothed rate. If the position data is represented by the angular displacement of a shaft in the computer, the quantity $E(0)$ is readily fixed by providing a second shaft which is coupled to the first shaft until $t = 0$ when the coupling is broken. Potentiometers mounted on the shafts are energized by a voltage varying as a function of time in the manner indicated in Figure 11. The manner in which the smoothed rate is obtained is clear.

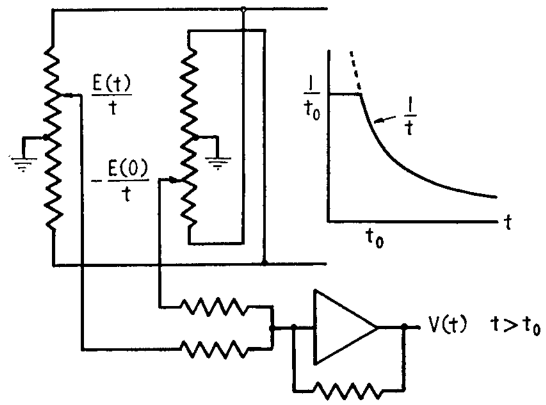


FIGURE 11. Memory point method of obtaining smoothed rate.

The memory point method of obtaining smoothed rates is used in the T15 antiaircraft director.[4] In this application, however, it is somewhat more complicated than in the simple illustration described above. This is due to the fact that the position data and the memory point are in the polar coordinate system, whereas the rate components are referred to a tilted and rotating rectangular coordinate system which is determined by the instantaneous line of sight.

Figure 12, shows a way of securing variable smoothing in a purely electrical circuit.[k] Except for the fact that the division of the current through the condensers is varied discontinuously instead of continuously, this circuit corresponds to the first or the second example discussed in Section 14.7.

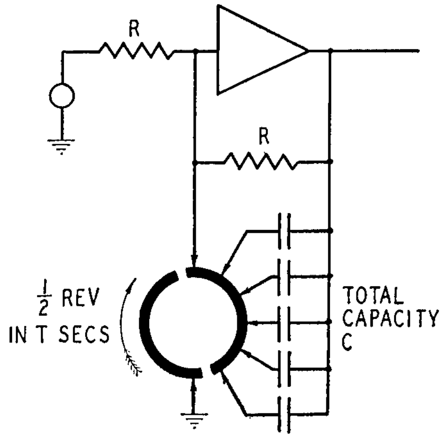


FIGURE 12. Specific limited range time-variable feedback smoothing circuit.

Figure 13 shows the variable smoothing circuit[l] for smoothing first derivatives in the M9A1-E1 antiaircraft director.[8] This circuit corresponds approximately to the second example of the differential equation (14) given above. The variable element is a thermistor which is heated up to a high temperature, practically instantaneously, by the heater, and then

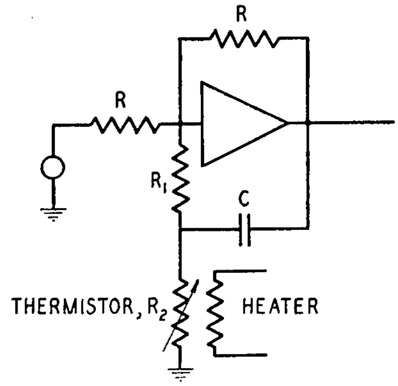


FIGURE 13. Another specific limited range time-variable feedback smoothing circuit.

[k] This circuit is due to S. Darlington.

[l] Developed by R. F. Wick.

allowed to cool off naturally. By choosing the electrical and thermal constants in the circuit correctly the resulting smoothing can be made to approximate that obtained in a memory point circuit.

As noted earlier, all these variable circuits require some auxiliary control means to reset the variable circuits to zero whenever a new target is engaged or the current target makes a sudden change in course. In the T15 memory point system this function was performed by an operator. The operator was aided by a series of meters which compared the instantaneous memory point rates with average rates set in some time previously by hand. The visual indication of a change in course, calling for the selection of a new memory point, was a relatively large, smoothly and decisively varying deflection on the meters. In contrast, normal tracking errors appeared as relatively small random fluctuations of the needles. The circuits of Figures 7 and 12, which were intended for bombsight applications, were also under the control of an operator, who was supposed to start the mechanism at the beginning of each bombing run.

Two control methods were used for the circuit of Figure 13. In one, large changes in rate, corresponding to probable changes in target course, were distinguished by comparing the instantaneous value of the target rate, as obtained directly from a differentiator, with the smoothed value obtained at the output of the smoothing circuit. In the other method, equivalent information was obtained by again differentiating the instantaneous value of the target rate, making a second derivative of the target coordinate. In either case this rate difference or second derivative information was used to control a gas tube, which went off, supplying heating current to the variable thermistor, whenever the voltage applied to it exceeded a certain threshold. This threshold evidently marks the minimum change in course for which the variable network will be reset. In order to permit the use of a low threshold, without making the circuit unduly liable to false operation because of the effect of tracking errors, the gas tube input voltage was first transmitted through a low-pass filter which suppressed most of the energy due to tracking errors. A considerable amount of work was done on the proportioning of this filter to provide the best protection against false operation with a low threshold and with minimum delay in resetting in case a change of course actually does occur, but the problem remains an interesting subject for research.

APPENDIX A

NETWORK THEORY

THIS APPENDIX GIVES a summary of linear network theory which is pertinent to the analysis and design of data-smoothing and prediction circuits. It is incomplete in many respects and should therefore be supplemented by reference to established textbooks on the subject. However, it contains some results which are new.

The present summary will be concerned mainly with fixed linear networks. Variable linear networks will be considered briefly in the last section.

A.1 IMPULSIVE ADMITTANCE

A fixed linear transmission network is one in which the response $V(t)$ is related to the impressed signal $E(t)$ by a linear differential equation of the form

$$b_n \frac{d^nV}{d(t)^n} + b_{n-1}\frac{d^{n-1}V}{d(t)^{n-1}} + \cdots + b_0 V = a_m \frac{d^mE}{d(t)^m} + a_{m-1}\frac{d^{m-1}E}{d(t)^{m-1}} + \cdots + a_0 E \quad (1)$$

with constant coefficients. It is well-known that the solutions of such a differential equation obey the "superposition principle." This makes it possible to formulate the response of the network to any signal, in terms of its response to certain standard signals.

A convenient standard signal for analytical purposes is the "unit impulse." It may be regarded as the limit of the rectangular pulse shown in Figure 1 as the duration of the pulse

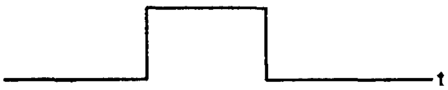


FIGURE 1. Rectangular pulse signal.

is decreased indefinitely while the amplitude is increased in such a way that the area under the pulse is always unity. The limiting function thus defined does not exist in a strict mathematical sense. However, it is very convenient for analytical purposes, and seldom leads to difficulties, to proceed as though the limiting function did exist. An impulse occurring at $t = \lambda$ is conventionally denoted by the singular function $\delta_0(t - \lambda)$ where

$$\begin{aligned} \delta_0(\tau) &\equiv 0 \quad \text{if } \tau \neq 0 \\ \int_{-\infty}^{t} \delta_0(\tau)d\tau &\equiv 0 \quad \text{if } t < 0 \\ &\equiv 1 \quad \text{if } t > 0 \end{aligned}$$

The response of a fixed network to an impulse or any form of signal is independent of the time at which the signal is applied, provided it is expressed as a function of the time relative to the application of the signal. Let $W(t)$ be the response to the signal $\delta_0(t)$. This is called the "impulsive admittance" of the network. Physically, it must be identically zero for negative values of t. For an impulse applied at $t = \lambda$ the response will therefore be $W(t - \lambda)$, which is identically zero for $t < \lambda$.

A physical signal $E(t)$ such as the one shown in Figure 2 may be resolved into an infinite

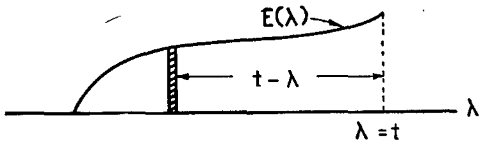


FIGURE 2. Derivation of superposition theorem.

succession of elementary impulses. The strength of the typical elementary impulsive component, such as the one shown in Figure 2 as occurring at time λ, is $E(\lambda)\,d\lambda$. Its contribution to the response at time t is $E(\lambda)\cdot W(t - \lambda)\,d\lambda$. Hence the contribution of all the elementary impulsive components of the signal, to the response at time t, is given by the formula

$$V(t) = \int_{-\infty}^{t+} E(\lambda) \cdot W(t - \lambda)d\lambda \quad (2)$$

This is one form of the "superposition theorem" for fixed linear networks.

Before discussing the reasons for the limits of integration indicated in (2), it will be helpful to consider a graphical interpretation other than the one used in deriving the integral. Let $W(t)$ be of the form shown in Figure 3, and let $E(\lambda)$ be of the form shown in Figure 4. To determine the response $V(t)$ at a given value of t, the curve in Figure 3 is turned over from right to left and placed over the curve in Figure 4 so that its right-hand edge is at $\lambda = t$. The product of the two curves gives a third curve (not shown), which is identically zero for all $\lambda > t$. The area under the third curve is the response $V(t)$ at the given value of t. For progressively larger values of t, the curve representing $W(t - \lambda)$ in Figure 4 is simply slid to the right with respect to the curve representing $E(\lambda)$.

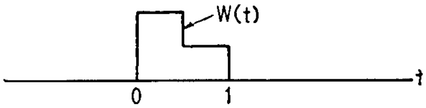


FIGURE 3. An illustrative impulsive admittance.

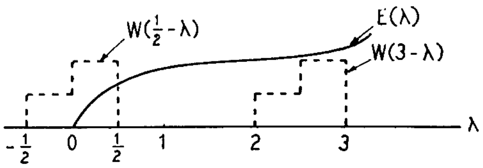


FIGURE 4. Graphical interpretation of superposition theorem.

Since a physical signal must certainly be identically zero up to some definite time, or since it must certainly have been applied to the network at some definite time, that time could be taken arbitrarily as zero and (2) could be written in the form

$$V(t) = \int_0^t E(\lambda) \cdot W(t - \lambda) d\lambda \tag{3}$$

In this form, however, since

$$\int_0^t W(t - \lambda) d\lambda = \int_0^t W(\tau) d\tau$$

is in general a function of t, the response could not be interpreted as a weighted average of the signal. On the other hand, since

$$\int_{-\infty}^t W(t - \lambda) d\lambda = \int_0^\infty W(\tau) d\tau$$

is independent of t, the response may be interpreted as a weighted average of the signal, if

$$\int_0^\infty W(\tau) d\tau = 1 \,.$$

The necessity of taking the lower limit in (2) as $-\infty$, in order to permit the interpretation of the response as a weighted average of the signal, is also expressed by the point of view that a fixed network cannot make any physical distinction between having no applied signal and having an applied signal which happens to be of zero amplitude.

Another shortcoming of the form (3) or, for that matter, of the form (2) if we set t as the upper limit of integration, comes from the consideration of impulsive admittances of such a nature that $W(t - \lambda)$ has certain kinds of singularities at $\lambda = t$. For example, the case for direct transmission, expressed in the form

$$V(t) = \int_{-\infty}^t E(\lambda) \cdot \delta_0(t - \lambda) d\lambda$$

is ambiguous because the singularity in the integrand occurs exactly at one end of the range of integration. However, the form

$$V(t) = \int_{-\infty}^{t+} E(\lambda) \cdot \delta_0(t - \lambda) d\lambda$$

leads, without ambiguity, to the result $V(t) = E(t)$. This example is not trivial. Every network which transmits infinite frequency must have an impulsive admittance of such a nature that $W(t - \lambda)$ contains a singularity of the form $\delta_0(t - \lambda)$. Any attempt to rule out such a singularity on the ground that physical networks cannot in fact transmit infinite frequency, complicates the analysis and design of networks unduly. If a network is capable of, or is expected to transmit frequencies at the top of the range of interest or importance, it is simpler to assume that the network is capable of, or is expected to transmit all frequencies above that range.

One other advantage of taking the limits of integration as indicated in (2) may be called to attention. Keeping in mind that $E(\lambda)$ is identically zero for all values of λ below some definite though perhaps unknown value, and that $W(t - \lambda)$ is identically zero for all values of $\lambda > t$, it is clear that (2) may be integrated partially any number of times without incurring the burden of carrying a string of terms outside of the integral. After one partial integration we have

$$V(t) = \int_{-\infty}^{t+} E'(\lambda) \cdot A(t - \lambda) d\lambda \tag{4}$$

where

$$A(t) = \int_{-\infty}^t W(\tau) d\tau \,. \tag{5}$$

Since $E'(\lambda)$ is identically zero for all values of λ in which $E(\lambda)$ is identically zero, and since

$A(t-\lambda)$ is identically zero for all values of $\lambda > t$, a second partial integration may be performed with no more formal complication than the first partial integration. The fact of the matter is that the terms which ordinarily arise in partial integrations, outside of the integral, are here carried under the integral by singularities of the integrand.

The superposition theorem in the form (4) may be derived directly in a manner similar to the derivation of (2). $A(t-\lambda)$ is the response of the network to a Heaviside unit step function $H(t-\lambda)$ applied at $t=\lambda$, where

$$\begin{aligned} H(t-\lambda) &\equiv 0 \quad \text{when } t < \lambda \\ &\equiv 1 \quad \text{when } t > \lambda \,. \end{aligned}$$

The signal is resolved into an infinite succession of elementary step functions of amplitude $E'(\lambda)d\lambda$ wherever $E(\lambda)$ is continuous, and finite step functions of amplitude $dE(\lambda)$ wherever $E(\lambda)$ has a finite discontinuity. The contribution of each elementary step function to the response at time t is $E'(\lambda)\cdot A(t-\lambda)d\lambda$, that of each finite step function is $A(t-\lambda)\cdot dE(\lambda)$. Hence, the response is given formally by (4) with the understanding that $E'(\lambda)d\lambda$ is to be interpreted as $dE(\lambda)$ wherever $E(\lambda)$ is discontinuous.[a]

The response $A(t)$ of the network to a Heaviside unit step function $H(t)$ applied at $t=0$ is called the "indicial admittance" of the network. It is more familiar, in the field of linear transmission theory, than the impulsive admittance to which it is related by (5), but in this monograph preference is given to the use of the impulsive admittance. In the theory of linear differential equations the impulsive admittance is known as a Green's function.

It is often convenient to express the response so that the variable of integration represents the age of the elementary components of the signal. Introducing the age variable

$$\tau = t - \lambda \tag{6}$$

into (2), we have

$$V(t) = \int_{0-}^{\infty} E(t-\tau)\cdot W(\tau)d\tau \,. \tag{7}$$

[a] Formula (4) may be written in the Stieltjes form

$$V(t) = \int_{-\infty}^{t+} A(t-\lambda)\, dE(\lambda).$$

Alternatively, we may take the point of view that $E'(\lambda)$ contains impulsive singularities wherever $E(\lambda)$ is discontinuous. This point of view is generalized in Appendix B.

In this form it is clear that the weighting of signal components is on the basis of age only. A fixed network may be said to have a memory which is a function only of the age of past events.

In the preliminary stages of designing a smoothing network, the weighting function $W(\tau)$ is generally prescribed to be identically zero when $\tau > T$ say, as well as when $\tau < 0$. This does not violate the conditions of physical realizability. However, such a weighting function cannot be obtained exactly with a network of a finite number of discrete impedance elements. A finite network invariably yields a weighting function with a "tail" which extends to infinity.

A.2 TRANSMISSION FUNCTION

Theoretically, the impulsive admittance of a prescribed network may be determined directly from the differential equations of the network in a perfectly straightforward manner. Practically, however, it is very difficult to do so if the network has more than two meshes. Furthermore, the technical problem of designing a network directly from a prescribed impulsive admittance is even more difficult, particularly if the impulsive admittance is not exactly realizable.

These difficulties may be avoided by recourse to the highly developed methods of network analysis and synthesis used in the field of communication circuits. These methods are based upon the steady-state properties of networks.

If a signal consisting of the single sinusoid $\cos \omega t$ is applied to an invariable or fixed linear transmission network, the steady-state response[b] will also be a single sinusoid of the same frequency. The amplitude and phase of the response, relative to the signal, will in general depend upon the frequency. The response may be regarded as the resultant of an "inphase component" proportional to $\cos \omega t$, and a "quadrature component" proportional to $\sin \omega t$, with amplitude coefficients which are functions of the frequency. Furthermore, since the signal is an even function of the frequency, the response should also be an even function of the frequency.[c] Hence, the response will

[b] This is the response apart from transient components, assuming that the latter vanish exponentially with time after the signal is impressed.

[c] The signal is also an even function of the time but this is due only to the particular choice of origin which is arbitrary.

be of the form $G(\omega^2)\cos\omega t - \omega H(\omega^2)\sin\omega t$, where G and H are even real functions of frequency.

By a suitable shift of the origin of time it follows that if the impressed signal is $\sin\omega t$, the steady-state response will be of the form $G(\omega^2)\sin\omega t + \omega H(\omega^2)\cos\omega t$.

These two results may be combined into a simpler expression without any loss of individuality. Since $e^{i\omega t} = \cos\omega t + i\sin\omega t$ where $i = \sqrt{-1}$, we have

$$V(t) = [G(\omega^2) + i\omega H(\omega^2)]\cdot e^{i\omega t} \quad \text{if } E(t) = e^{i\omega t}.$$

A further simplification may be achieved by replacing $i\omega$ by p, and $G(-p^2) + pH(-p^2)$ by $Y(p)$, so that

$$V(t) = Y(p)\cdot e^{pt} \quad \text{if } E(t) = e^{pt}. \tag{8}$$

$Y(p)$ is called the "steady-state transmission function" or just "transmission function" for short.

Strictly speaking, (8) expresses the relation of steady-state response to signal only if $p = i\omega$. However, it is customarily called a steady-state relation even when p is not a pure imaginary quantity. It may be noted that $Y(p)$ is real when p is real.

The simplicity of steady-state analysis derives from the fact that time occurs in the signal and throughout the network only in the form e^{pt}. In particular, the determination of the transmission function is reduced to the solution of simultaneous algebraic equations which do not involve the time factor. For a network in which the signal and the response are related by the linear differential equation (1) with constant coefficients, we obtain simply

$$Y(p) = \frac{a_0 + a_1p + \cdots + a_mp^m}{b_0 + b_1p + \cdots + b_np^n}.$$

It may be noted that the poles of the transmission function, also referred to as "infinite-gain points" in the p-plane, correspond to the roots of the characteristic function of the differential equation. Physical restrictions on the location of infinite-gain points will be considered in Section A.9.

A.3 RELATIONSHIP BETWEEN IMPULSIVE ADMITTANCE AND TRANSMISSION FUNCTION

A relationship between the impulsive admittance and the transmission function of a network may be obtained from (7). Putting $E(t) = e^{pt}$ when $t > 0$, we get

$$\begin{aligned} V(t) &= e^{pt}\int_{0-}^{t} W(\tau)\, e^{-p\tau}\, d\tau \\ &= e^{pt}\int_{0-}^{\infty} W(\tau)\, e^{-p\tau}\, d\tau \\ &\quad - e^{pt}\int_{t}^{\infty} W(\tau)\, e^{-p\tau}\, d\tau . \end{aligned} \tag{9}$$

The second term in (9) is a transient term due to the fact that we have taken $E(t) \equiv 0$ when $t < 0$. The first term in (9), which involves the time only through e^{pt}, is the steady-state term. Comparing this term with (8) we get

$$Y(p) = \int_{0-}^{\infty} W(t)\, e^{-pt}\, dt \tag{10}$$

or, in the notation which will be introduced in the next section

$$Y(p) = L[W(t)]. \tag{11}$$

A.4 LAPLACE AND INVERSE LAPLACE TRANSFORMS

The frequent use which is made of the Laplace transform and its inverse, in the analysis and design of fixed linear networks, warrants a brief discussion of these transforms.

Given a function $f(t)$ which is identically zero when $t < 0$, its Laplace transform $g(p)$ is defined by the formula

$$g(p) = L[f(t)] = \int_{0-}^{\infty} f(t)\, e^{-pt}\, dt. \tag{12}$$

This is usually written with 0 for the lower limit, but by having the point $t = 0$ inside the range of integration, instead of at the end, we secure the same advantages for (12) that we gained in the case of (2) by having the point $\lambda = t$ inside the range of integration. Since $f(t)$ is identically zero when $t < 0$ we could write $-\infty$ for the lower limit in (12), but this would run the risk of confusion with the so-called "bilateral Laplace transform." On the whole, it is worth while to have a constant reminder that functions $f(t)$ which are not identically zero when $t < 0$ are ruled out.

The integral in (12) is usually not convergent for all values of p. That is, in order to secure convergence of the integral, it may be necessary to assume $R(p) > a$, where $R(p)$ is the real part of p, and a is a real number. The result of the integration is a representation of $g(p)$ in the half-plane $R(p) > a$. Since the representation is analytic throughout the half-plane, the principle of analytic continuation allows us to extend the definition of $g(p)$ to the remainder of the p-plane.

Given a function $g(p)$ which is analytic throughout the half-plane $R(p) \geq c$ where c is a real number, its inverse Laplace transform $f(t)$ is given by the formula

$$f(t) = L^{-1}[g(p)] = \frac{1}{2\pi i} \int_{c+i\infty}^{c+i\infty} g(p)\, e^{pt}\, dp \qquad (13)$$

provided $f(t)$ is identically zero when $t < 0$. If the result of the integration in (13) is not identically zero when $t < 0$, $g(p)$ is not a Laplace transform and the application of the inverse transformation to it is meaningless.

Translation Theorem

A useful theorem can be established at this point. This is the *translation theorem.*
If

$$G(p) = L[F(t)]$$

then

$$L^{-1}[G(p)\, e^{-pa}] = F(t-a)$$

provided that $F(t-a) \equiv 0$ when $t < 0$. Translation is to the right or left according as a is positive or negative.

If it happens that $F(t) \equiv 0$ when $t < t_0$ where $t_0 \geq 0$, then the restriction is that $a \geq -t_0$. That is, a limited amount of translation to the left is permissible. In general, $t_0 = 0$ and the restriction is therefore that $a \geq 0$. This theorem follows readily from (12) or (13).

In all of the applications of (13) which we have any occasion to make in the analysis and design of fixed linear networks, the function $g(p)$ may be resolved into a sum of terms of the form $G(p)e^{-pa}$ where $a \geq 0$ and $G(p)$ is a rational algebraic function with real coefficients. Making use of the translation theorem, the problem of evaluating $L^{-1}[g(p)]$ reduces to that of evaluating $L^{-1}[G(p)]$. Now, $G(p)$ may be resolved into a sum of terms of the form p^m or $1/(p-\alpha)^{m+1}$ where $m = 0, 1, 2 \cdots$. We shall consider these two cases separately.

The case $G(p) = p^m$ will be treated by means of (12) and some limiting processes. In Section A.1 the unit impulse was regarded as the limit of a rectangular pulse of duration T and amplitude $1/T$. By means of (12) the Laplace transform of such a pulse over the interval $0 \leq t \leq T$ is

$$\frac{1 - e^{-pt}}{pT}$$

Hence

$$L[\delta_0(t)] = \lim_{T \to 0} \frac{1 - e^{-pT}}{pT} = 1 .$$

Formally therefore

$$L^{-1}[1] = \delta_0(t) . \qquad (14)$$

Similarly, the Laplace transform of a pulse over the interval $a \leq t \leq a + T$ where $a > 0$ is

$$\frac{1 - e^{-pT}}{pT} e^{-pa}$$

Hence

$$L[\delta_0(t-a)] = \lim_{T \to 0} \frac{1 - e^{-pT}}{pT} e^{-pa} = e^{-pa} .$$

Formally therefore

$$L^{-1}[e^{-pa}] = \delta_0(t-a) .$$

The last result follows directly from (14) using the translation theorem.

Next, let

$$\delta_1(t) = \lim_{T \to 0} \frac{\delta_0(t) - \delta_0(t-T)}{T} .$$

This is the limiting case, as shown in Figure 5, of two impulses of strengths $1/T$ and $-1/T$ separated by a time interval T. It may be called

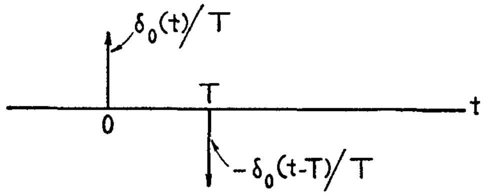


FIGURE 5. An impulse doublet.

an impulse of second order. By (12) and the previous results

$$L[\delta_1(t)] = \lim_{T \to 0} \frac{1 - e^{-pT}}{T} = p .$$

Formally therefore

$$L^{-1}[p] = \delta_1(t) . \qquad (15)$$

Proceeding in this fashion we may define an impulse of $(m+1)$th order as

$$\delta_m(t) = \lim_{T \to 0} \frac{\delta_{m-1}(t) - \delta_{m-1}(t-T)}{T} \qquad (16)$$

and we may then show that

$$L[\delta_m(t)] = p^m .$$

Formally therefore

$$L^{-1}[p^m] = \delta_m(t) . \tag{17}$$

This disposes of the case $G(p) = p^m$ where $m = 0, 1, 2 \cdots$.

The case $G(p) = 1/(p - \alpha)^{m+1}$ will be treated by means of (13) and Jordan's lemma.

JORDAN'S LEMMA

If all the singularities of $G(p)$ can be enclosed by a circle of finite radius with center at the origin, and if $G(p) \to 0$ uniformly with respect to arg z as $|z| \to \infty$, then

$$\lim_{\rho \to \infty} \left[\int_\Gamma G(p) e^{pt} \, dp \right] = 0$$

where Γ is a semicircle of radius ρ, with center at the origin, to the right of the imaginary axis if t is negative, to the left of the imaginary axis if t is positive.

By the use of this lemma the contour of integration in (13) may be closed and the integration may then be performed by the method of residues. In the case

$$G(p) = \frac{1}{(p - \alpha)^{m+1}} \quad \text{where } m = 0, 1, 2 \cdots$$

we readily obtain

$$L^{-1}\left[\frac{1}{(p - \alpha)^{m+1}}\right] \equiv 0 \qquad t < 0$$
$$= \frac{t^m e^{\alpha t}}{m!} \qquad t > 0 . \tag{18}$$

An important special case of (18), corresponding to $\alpha = 0$, is

$$L^{-1}\left[\frac{1}{p^{m+1}}\right] = \frac{t^m}{m!} \qquad t > 0 . \tag{19}$$

Another useful theorem which is readily established by means of (12) and (13) is Borel's theorem.

BOREL'S THEOREM

If $g(p)$, $g_1(p)$, $g_2(p)$ are the Laplace transforms of $f(t)$, $f_1(t)$, $f_2(t)$, respectively, and if

$$g(p) = g_1(p) \, g_2(p)$$

then

$$f(t) = \int_{0-}^{t+} f_1(t - \lambda) \cdot f_2(\lambda) \, d\lambda$$
$$= \int_{0-}^{t+} f_1(\tau) \cdot f_2(t - \tau) \, d\tau .$$

The functions $f_1(t)$ and $f_2(t)$ are subject to conditions which permit the inversion of the order of integration in the following proof. However, these conditions are seldom of any concern. We have

$$f(t) = L^{-1}\{g_1(p) \cdot L[f_2(t)]\}$$
$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[g_1(p) e^{pt} \int_{0-}^{\infty} f_2(\lambda) e^{-p\lambda} \, d\lambda \right] dp .$$

Inverting the order of integration and noting that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g_1(p) e^{p(t-\lambda)} \, dp = 0 \qquad \text{if } \lambda > t$$
$$= f(t - \lambda) \qquad \text{if } \lambda < t$$

we obtain the result stated in the theorem.

A.5 ALTERNATIVE EXPRESSION OF THE RESPONSE-TO-SIGNAL RELATIONSHIP

The result (8) obtained in Section A.2 suggests an operational expression of the form

$$V(t) = Y(p) \cdot E(t) \tag{20}$$

for the response-to-signal relationship whatever the signal $E(t)$ might be. If the equivalence of this operational expression to (2) is taken as a matter of definition we may readily discover the nature of the implied operation.

In the light of Borel's theorem, (2) may be expressed in the form

$$L[V(t)] = L[W(t)] \cdot L[E(t)]$$

under the permissible assumption that $E(t) \equiv 0$ when $t < 0$. Hence

$$V(t) = L^{-1}\{L[W(t)] \cdot L[E(t)]\}$$

or, by (11)

$$V(t) = L^{-1}\{Y(p) \cdot L[E(t)]\} . \tag{21}$$

This is, therefore, in general the meaning of the operational expression (20).[d]

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[d] We note that if $S(p) = L[E(t)]$, the operational expression

$$V(t) = S(p) \cdot W(t)$$

is equivalent to (20). This form is used in Section 10.4 and in Appendix B.

In very special cases of $Y(p)$ it is possible and useful to give a more direct meaning to the operation (20). Consider the case $Y(p) = p$. We have, from the preceding section

$$W(t) = L^{-1}[p] = \delta_1(t)$$
$$= \lim_{T\to 0} \frac{\delta_0(t) - \delta_0(t-T)}{T}.$$

Substituting into (7) we get

$$V(t) = \lim_{T\to 0} \frac{E(t) - E(t-T)}{T} = \frac{dE(t)}{dt}.$$

Hence, p stands for the derivative operator d/dt. Higher powers of p correspond to derivatives of higher orders because of (17) and (16).

For the case $Y(p) = 1/p$ we have, from the preceding section

$$W(t) = L^{-1}[p^{-1}] = 0 \quad \text{if} \quad t < 0$$
$$= 1 \quad \text{if} \quad t > 0$$

substituting into (2) we get

$$V(t) \int_{-\infty}^{t} E(\lambda)\, d\lambda.$$

Hence, negative integral powers of p correspond to multiple integrations from $-\infty$ to t.

These results are applicable to relations between network characteristics. Suppose, for example, that

$$Y(p) = p^m y(p)$$

and that

$$W(t) = L^{-1}[Y(p)] \text{ and } w(t) = L^{-1}[y(p)].$$

Then

$$W(t) = \frac{d^m}{dt^m} w(t)$$

and conversely $w(t)$ is the m-fold integral of $W(t)$ from $-\infty$ to t.

Another special case of $Y(p)$ to which a more direct operational meaning may be given is e^{-pa} where $a \geq 0$. In view of the translation theorem we get

$$V(t) = E(t - a)$$

That is, the response is a retarded facsimile of the signal. This result is to be expected on the ground that e^{-pa} corresponds to the transmission function of a properly terminated distortionless, nondissipative, uniform transmission line with delay a.

PERFECT PREDICTION OPERATOR

The result obtained in the last paragraph suggests e^{pt_f} as a perfect prediction operator, where t_f is the time of flight, since it gives the result that

$$V(t) = E(t + t_f) \quad \text{if} \quad t > -t_f$$

Since it implies that a signal will be received before it is sent, however, it cannot be physically realizable. It is nevertheless useful as an analytical tool in cases in which the signal has continuous derivatives of all orders in the closed interval t to $t + t_f$, such as in the steady-state case considered in Section 13.1.4.

A.6 TANDEM NETWORKS

The transmission function $Y(p)$ of a tandem combination of two networks whose individual transmission functions are $Y_1(p)$ and $Y_2(p)$, respectively, is given by

$$Y(p) = Y_1(p) \cdot Y_2(p). \tag{22}$$

By Borel's theorem, therefore, the impulsive admittance of the combination is given in terms of the individual impulsive admittances by

$$W(t) = \int_{0-}^{t+} W_1(t - \lambda) \cdot W_2(\lambda)\, d\lambda. \tag{23}$$

The advantages of methods of network analysis and synthesis based upon steady-state properties are due partly to the essential simplicity of (22) compared with (23). These advantages are particularly important in network synthesis or design. In this case $Y(p)$ is a prescribed rational function of p. It is a simple matter to resolve it into two or more factors. A little experience in network design can go a long way toward a choice of factors which is the most favorable, on the whole, from the point of view of network configurations (including feedback networks) and element values. Redundant factors are easily introduced if they are desirable, as is done in the practical designs described in Section 13.1.5 and Section 13.3.

A.7 SYMMETRICAL IMPULSIVE ADMITTANCES

Symmetrical impulsive admittances occur very frequently in the theory of data-smoothing networks. (See Section B.2.) The transmission functions corresponding to them possess a property which we shall bring out in this section.

The symmetry of the impulsive admittance is expressed by

$$W(T - t) = W(t)$$

Since $W(t) \equiv 0$ when $t < 0$, it must be so also when $t > T$. Hence

$$Y(p) = \int_{0-}^{T/2} W(t)e^{-pt}\,dt + \int_{T/2}^{T+} W(t)e^{-pt}\,dt.$$

By a change of variable of integration the second term may be expressed in the form

$$\int_{0-}^{T/2} W(T-t)e^{-p(T-t)}\,dt$$

or, because of the symmetry, in the form

$$e^{-pT}\int_{0-}^{T/2} W(t)e^{pt}\,dt\,.$$

Hence, if the first term in $Y(p)$ be denoted by

$$Y_1(p) = \int_{0-}^{T/2} W(t)e^{-pt}\,dt$$

we have

$$\begin{aligned} Y(p) &= Y_1(p) + Y_1(-p)\,e^{-pT} \\ &= [Y_1(p)e^{pT/2} + Y_1(-p)e^{-pT/2}]\,e^{-pT/2}\,. \end{aligned}$$

At real frequencies ($p = i\omega$) the bracketed factor is evidently an even real function of ω. Hence

$$Y(i\omega) = Q(\omega^2)\cdot e^{-i\omega T/2}. \qquad (24)$$

Apart from discontinuities in the phase angle of the transmission function at real frequencies ω for which $Q(\omega^2)$ is zero, the phase angle is proportional to frequency. Such a transmission function is referred to as a linear phase transmission function. Sinusoidal components of the signal, of frequencies less than the lowest frequency at which $Q(\omega^2)$ vanishes, suffer phase retardations in transmission in proportion to their frequencies. These components therefore contribute no delay distortion. They are delayed by a uniform amount, just as they are in a properly terminated distortionless, uniform transmission line, although in the case of (24) they contribute amplitude or loss distortion through $Q(\omega^2)$. The delay in (24) is just half of the "smoothing time" T.

A.8 SERIES RELATIONSHIPS BETWEEN IMPULSIVE ADMITTANCE AND TRANSMISSION FUNCTION

Two useful series relationships between impulsive admittances and transmission functions will be derived in this section.

Assume that $W(t)$ admits the series expansion

$$W(t) = A_0 + A_1 t + \cdots + \frac{A_m t^m}{m!} + \cdots. \qquad (25)$$

for small positive values of t. Then by **(11)** and (19)

$$Y(p) = \frac{A_0}{p} + \frac{A_1}{p^2} + \cdots + \frac{A^m}{p^{m+1}} + \cdots. \qquad (26)$$

If $A_0 \neq 0$ the transmission cannot drop off faster than 6 db per octave as the frequency increases indefinitely. If the transmission is to drop off ultimately at the rate of $6k$ db per octave all of the A's up to and including A_{k-2} must be zero. This is to say that the impulsive admittance and all of its derivatives of orders up to and including the $(k-2)$th must vanish at $t = 0$.

Next, let us suppose that the impulsive admittance and all of its derivatives of orders up to and including the $(k-2)$th are continuous through all values of t including $t = 0$ except that the $(k-2)$th derivative is discontinuous only at $t = a$. We may resolve the impulsive admittance into the sum $W_1(t) + W_2(t)$ where $W_1(t)$ and all of its derivatives of orders up to and including the $(k-2)$th are continuous through all values of t including $t = 0$, while $W_2(t) \equiv 0$ for all values of $t < a$. Then, for small positive values of $t - a$

$$W_2(t) = \frac{A_{k-2}(t-a)^{k-2}}{(k-2)!} + \cdots \qquad (A_{k-2} \neq 0)$$

whence

$$Y_2(p) = \left(\frac{A_{k-2}}{p^{k-1}} + \cdots\right)e^{-pa}\,.$$

Hence the transmission cannot drop off ultimately faster than $6(k-1)$ db per octave. We may summarize these results in the asymptotic loss theorem.

Asymptotic Loss Theorem.

If the transmission is to drop off ultimately at the rate of $6k$ db per octave as the frequency increases indefinitely, the impulsive admittance and all of its derivatives of orders up to and including the $(k-2)$th must be continuous through all values of t including $t = 0$.

Discontinuities in $W(t)$ or in some derivative of $W(t)$ cannot occur except at $t = 0$ in the case of physical lumped element networks. Practically, however, rapid changes in $W(t)$

or in some derivative of $W(t)$, at any value of t, may be expected to be associated with much the same behavior of the transmission at reasonably high frequencies. As an example consider the case

$$W(t) = e^{-\alpha t} - e^{-\beta t} \qquad (\beta > \alpha > 0)$$

$$Y(p) = \frac{\beta - \alpha}{(p + \alpha)(p + \beta)} .$$

$W(t)$ is continuous through $t = 0$ as long as β is finite but becomes discontinuous there in the limit as $\beta \to \infty$. The first derivative of $W(t)$ is discontinuous through $t = 0$ even when β is finite. The ultimate slope of the transmission is 12 db per octave, in accordance with the asymptotic loss theorem, but in the range $\alpha < \omega < \beta$ the transmission appears to have a slope of only 6 db per octave.

The importance of the observations made in the preceding paragraph, in the design of a network, is that if we attempt to approximate a $W(t)$ which has a discontinuity in a derivative of lower order at $t = a$ than at $t = 0$, the fact that the physical approximation must have continuous derivatives of all orders and through all values of t except $t = 0$ is not very significant. The ultimate slope of the transmission may not be reached until the frequency is too high to be of any importance.

Another useful relationship between impulsive admittance and transmission function follows from the assumption that $\int_{0-}^{\infty} t^m W(t)\, dt$ is finite for $m = 0, 1, 2 \ldots$. If we expand the exponential in

$$Y(p) = \int_{0-}^{\infty} W(t) e^{-pt}\, dt$$

into a power series in pt we get

$$Y(p) = M_0 - M_1 p + \frac{M_2 p^2}{2!} - \frac{M_3 p^3}{3!} + \cdots \qquad (27)$$

where

$$M_m = \int_{0-}^{\infty} t^m W(t) dt . \qquad (28)$$

The quantity M_m is the mth moment of the impulsive admittance.

When $M_0 = 1$ we speak of the response of the network as a weighted average of the impressed signal, and speak of the impulsive admittance $W(t)$ as the weighting function.

A.9 PHYSICAL RESTRICTIONS ON THE TRANSMISSION FUNCTION

The transmission function $Y(p)$ of a lumped element network is a rational algebraic function of p. It is real for real values of p (A.2). Hence, the coefficients must be real, and therefore the roots and poles must either be real or occur in conjugate complex pairs.

Such a function may be expanded into the sum of a polynomial and a rational function whose numerator is of lower degree than the denominator. The latter may therefore be properly expanded into partial fractions. For a partial fraction of the form

$$\frac{1}{(p - \alpha)^m} \qquad \text{where } m = 1, 2 \cdots$$

the contribution to the impulsive admittance $W(t)$ is by (18)

$$L^{-1}\left[\frac{1}{(p - \alpha)^m}\right] = \frac{t^{m-1}}{(m - 1)!} e^{t} \qquad (t > 0) .$$

For a pair of partial fractions of the form

$$\frac{A + iB}{(p - \alpha + i\beta)^m} + \frac{A - iB}{(p - \alpha - i\beta)^m}$$

the contribution to the impulsive admittance is

$$\frac{2t^{m-1}}{(m - 1)!} e^{\alpha t} (A \cos \beta t + B \sin \beta t) .$$

Since the impulsive admittance is the response to an impulsive signal it is clear that for a stable network the impulsive admittance must be free of terms which increase indefinitely with time, either on account of an amplitude factor of the form $e^{\alpha t}$ where $a > 0$, or, in the event that $\alpha = 0$, on account of an amplitude factor of the form t^{m-1} where $m > 1$. Hence, the physical restrictions on the transmission function are:

1. No poles with positive real parts.
2. Poles on the imaginary p axis must be simple.[e]

The poles of a passive transmission function correspond to modes of free motion.[15h] Each of them may be shown[15i] to satisfy an equation of the form

$$pT + F + \frac{V}{p} = 0$$

where T, F, V are positive quantities whose values depend upon the particular mode and

[e] Poles on the imaginary p axis must also be ruled out on the ground that persistent transients cannot be tolerated any more than growing transients.

its activity. However, T is zero in the absence of kinetic energy, F is zero in the absence of energy dissipation, and V is zero in the absence of potential energy. It follows that in the absence of coils or in the absence of condensers, the transmission function must have poles only on the negative real p axis.

For extremely narrow-band, low-pass applications, such as data smoothing, it is not practicable to build networks which call for coils because these generally turn out to be of many thousands of henries in inductance. The exclusion of coils from these applications does not, however, rule out transmission functions with complex poles. These may be realized with RC networks in feedback amplifier circuits as is shown in Chapter 12.

A.10 QUASI-DISTORTIONLESS TRANSMISSION NETWORKS

A quasi-distortionless transmission network is one which is distortionless only in a certain sense. This sense will be made clear in this section.

Let

$$Y(p) = \frac{1 + a_1 p + a_2 p^2 + \cdots + a_m p^m}{1 + b_1 p + b_2 p^2 + \cdots + b_n p^n}. \quad (29)$$

This may also be written in the form

$$Y(p) = 1 + c_1 p + \frac{c_2 p^2}{2!} + \cdots + \frac{c_r p^r}{r!} + p^{r+1} g(p). (30)$$

Obviously $g(p)$ will be a rational function with the same denominator as $Y(p)$ and a numerator of $(n-1)$th degree. If we now apply a signal of the form

$$\begin{aligned} E(t) &= 0 \quad \text{for } t < 0 \\ &= t^r \quad \text{for } t > 0 \end{aligned}$$

the response, by (21), will be

$$V(t) = t^r + rc_1 t^{r-1} + \frac{r!}{2!(r-2)!} c_2 t^{r-2} + \cdots + c_r + r!\, L^{-1}[g(p)] \qquad (t > 0).$$

If the coefficients in the rational expression for $Y(p)$ are such that

$$c_1 = t_f,\ c_2 = t_f^2, \cdots c_r = t_f^r \quad (31)$$

then

$$V(t) = (t + t_f)^r + r!\, L^{-1}[g(p)] \qquad (t > 0). \quad (32)$$

The second term vanishes exponentially with time. The first term is an advanced or a retarded facsimile of the applied signal according to whether t_f is positive or negative. We shall say that $Y(p)$ is the transmission function of a network which is quasi-distortionless to the signal t^r.

Obviously a transmission network which is quasi-distortionless to the signal t^r must also be quasi-distortionless to every signal t^s where s is a positive integer less than r, including zero. Hence we may state the quasi-distortionless transmission theorem.

Quasi-Distortionless Transmission Theorem

If the signal

$$\begin{aligned} E(t) &= 0 \text{ for } t < 0 \\ &= \text{polynomial of degree } r \text{ at most in } t \text{ for } t > 0 \end{aligned}$$

is applied to a "quasi-distortionless transmission network of order r," the response will be of the form

$$V(t) = E(t + t_f) + 0(e^{-t}) \qquad \text{for } t > 0,$$

where $0(e^{-t})$ stands for terms which vanish exponentially with time.

If $t_f > 0$ the transmission network is a predictor for polynomials of degree r at most. However, it does not begin to predict properly until some time has elapsed after the start of the signal, or of a new analytic segment of the signal; that is, until the transients have subsided sufficiently.

If $t_f = 0$ the transmission network may be regarded as a delay-corrected smoother for polynomials of degree r at most. This is obtained simply by taking

$$a_1 = b_1,\ a_2 = b_2, \cdots a_r = b_r \quad (33)$$

in (29).

A.11 VARIABLE LINEAR NETWORKS

A variable linear transmission network is one in which the response $V(t)$ is related to the impressed signal $E(t)$ by the linear differential equation (1) with coefficients which are prescribed functions of t. The solutions of such a differential equation also obey the superposition principle. Thus it is possible in this case also to formulate the response of the network to any signal in terms of its response to a standard impulsive signal.

The response of a variable network to an impulse or any form of signal depends, however, on the time at which the signal is applied. For an impulsive signal applied at time λ the response at time t will be represented by $W(t,\lambda)$. This is still called the "impulsive admittance." In the theory of linear differential equations it is known as a Green's function. Physically, it must be identically zero for $t < \lambda$.

The superposition theorem may now be written in the form

$$V(t) = \int_{0-}^{t+} E(\lambda) \cdot W(t,\lambda)\, d\lambda \qquad (34)$$

provided the network has been properly designed and set into operation at $t = 0$. If

$$\int_{0-}^{t+} W(t,\lambda)\, d\lambda = 1$$

for all values of $t > 0$, the response may be interpreted as a weighted average of the signal. We note that in order to interpret the response as a weighted average of the signal, it is now no longer necessary to take the lower limit in (34) as $-\infty$, as it was in the case of (2) for a fixed network. In other words, a variable network can be designed and set into operation at any time so that components of the signal which arrive before that time are completely ignored.

The analysis and design of variable linear networks are in general much more difficult than those of fixed linear networks. This is due largely to the fact that there does not yet exist a technique corresponding to the steady-state and operational methods used in connection with fixed networks. However, there is a class of variable networks whose analysis and design are greatly facilitated by the fact that they are related to fixed networks by a transformation of the time variable.

Consider the linear differential equation

$$b_n \frac{d^n V}{dz^n} + b_{n-1} \frac{d^{n-1} V}{dz^{n-1}} + \cdots + b_1 \frac{dV}{dz} + V = E$$

with constant coefficients. With appropriate restrictions on the roots of the characteristic function

$$b_n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_1 \lambda + 1$$

it represents the response-to-signal relationship in a fixed network, if z is proportional directly to time. However, if z is a more general function of the time, it will correspond to a variable network. The kind of transformation which is desired here is one which transforms the range $-\infty < z < +\infty$ into the range $0 < t < +\infty$ with a one-to-one correspondence. Thus, we may take $z = \log \theta(t)$ where $\theta(t)$ is a positive monotonic increasing function of t in the range $0 < t < +\infty$, with $\lim_{t \to 0} \theta(t) = 0$. Several examples of $\theta(t)$, including $\theta(t) \equiv t$, are considered in detail in Chapter 14.

APPENDIX B

THEORETICAL MODIFICATIONS OF SMOOTHING FUNCTIONS TO FIT NONUNIFORM NOISE SPECTRA

Best smoothing or weighting functions have been determined in Chapters 10 and 11 under the assumption of random noise with flat spectrum. It has not been worth while in practice to base the choice of best weighting functions on any more elaborate considerations of actual noise spectra, for at least three reasons:

1. The effectiveness of a smoothing network shape of the weighting function.
2. Noise spectra are subject to variations, due to factors which it is not desirable in practice to attempt to control.
3. Elaborate smoothing functions require elaborate networks with close tolerances on element values.

Nevertheless, the theory of smoothing presented in this monograph would not be complete without showing how more general shapes of noise spectra can be considered. Two methods are presented here, which are generalizations of those presented in Sections 10.3 and 10.4, respectively.

B.1 PHILLIPS AND WEISS THEORY[7]

Let $g(t)$ be the tracking error, and $W(t)$ the impulsive admittance of a smoothing and prediction circuit with smoothing time T. Then the error in prediction due to tracking error only, is

$$V(t) = \int_0^T g(t-\tau)\cdot W(\tau)\,d\tau.$$

The impulsive admittance $W(\tau)$ will depend also upon the time of flight which, for purposes of analysis, is assumed to be constant. The mean square error is then

$$\overline{V^2} = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} V^2(t)\,dt$$

$$= \int_0^T\int_0^T W(\tau_1)\cdot C(\tau_1-\tau_2)\cdot W(_2)d\tau_1 d\tau_2$$

where

$$C(x) = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} g(\lambda)\cdot g(\lambda+x)\,d\lambda\;\cdot \qquad (1)$$

$C(x)$ is the autocorrelation of the error time-function $g(\lambda)$.

For an nth order smoothing and prediction circuit $\overline{V}^2$ is now minimized with respect to the impulsive admittance under the restrictions[a]

$$\int_0^T \tau^m W(\tau)d\tau = (-t_f)^m \quad (m = 0, 1, 2 \cdots n). \qquad (2)$$

Hence $W(\tau)$ must satisfy the integral equation

$$\int_0^T C(t-\tau)\cdot W(\tau)d\tau = k_0 + k_1 t + \cdots + k_n t^n \qquad (0 \le t \le T)$$

where the k_m are constants to be determined. Now, if

$$\int_0^T C(t-\tau)\cdot W_m(\tau)d\tau = t^m \;(0 \le t \le T) \qquad (m = 0, 1, 2 \cdots n) \qquad (3)$$

then

$$W(\tau) = k_0W_0(\tau) + k_1W_1(\tau) + \cdots + k_nW_n(\tau). \qquad (4)$$

The procedure is then to determine $C(x)$ from (1), the $W_m(\tau)$ from (3), the k_m from (2) and (4), and finally $W(\tau)$ from (4). It may be noted that, in general, every k_m will be a polynominal of nth degree in t_f. Hence the $W_m(\tau)$ appearing here are not the same as those defined in Chapter 11, although $W(\tau)$ should be the same if the same $W_0(\tau)$ is used in Chapter 11.

A difficulty of the theory given above is in the solution of the integral equations (3). This difficulty is avoided in the theory given in the next section. However, the integral equations are easily solved in case of flat random noise, when $C(x)$ is simply an impulse of strength K say, at $x = 0$. Then

$$W_m(\tau) = \frac{\tau^m}{K} \qquad 0 < \tau < T.$$

Since the strength is irrelevant, it may be taken equal to T so that $W_0(\tau)$ will be normalized.

[a] These follow from the discussions in Sections A.8 and A.10, especially equations (27), (28), (30), and (31).

For a linear prediction circuit it is then found that

$$W(\tau) = 2\left(2 + \frac{3t_f}{T}\right)W_0(\tau) - \frac{6}{T}\left(1 + \frac{2t_f}{T}\right)W_1(\tau).$$

Putting $T = 1$ this may be expressed as

$$W(\tau) = w_0(\tau) + G_1(-t_f)w_1^{(1)}(\tau)$$

in terms of the $G_m(\tau)$ and $W_m(\tau)$ of Section 11.3.

B.2 SYMMETRY OF BEST SMOOTHING FUNCTIONS

The theory of Phillips and Weiss offers the most direct proof that the best smoothing or weighting function must be symmetrical, regardless of the noise power spectrum. The situation is that of minimizing (1) under only one of the restrictions (2), viz., the normalizing condition

$$\int_T^0 W(\tau)d\tau = 1 \tag{5}$$

The weighting function is therefore determined, up to a constant scale factor, by the condition that

$$\int_0^T C(t-\tau)\cdot W(\tau)d\tau = k, \tag{6}$$

where k is a constant. Substituting $T - t$ for t and $T - \tau$ for τ, we have

$$\int_0^T C(\tau - t)\cdot W(T-\tau)d\tau = k. \tag{7}$$

Since $C(-x) = C(x)$, and since $W(\tau)$ is determined uniquely by (6) and (5), it follows from (6) and (7) that

$$W(T-\tau) = W(\tau). \tag{8}$$

B.3 GENERALIZATION OF ELEMENTARY PULSE METHOD

The noise power transmitted through a network may be expressed in the familiar form

$$P = \int_0^\infty N(\omega^2)\cdot|Y(i\omega)|^2 d\omega$$

where $N(\omega^2)$ is the noise power spectrum and $Y(p)$ is the transmission function of the network. Assuming that $N(\omega^2)$ is a rational function of ω^2, which is finite at all finite values of ω including zero, it is possible to determine a rational function $S(p)$, which has no poles on or to the right of the imaginary axis in the p-plane with the exception of the point at infinity, and such that

$$|S(i\omega)|^2 = N(\omega^2).$$

It may be readily shown that

$$P = \pi\int_{0-}^\infty [F(t)]^2 dt \tag{9}$$

where $F(t)$ is related to the impulsive admittance $W(t)$ by the operational equation

$$F(t) = S(p)\cdot W(t) \tag{10}$$

The problem is now to minimize (9) under the restriction

$$\int_{0-}^{t_0} W(t)dt = 1 \text{ when } t_0 > 1. \tag{11}$$

Let

$$S(p) = k\,\frac{Q(p)}{R(p)}$$

where

$$Q(p) = (p+\alpha_1)(p+\alpha_2)\cdots(p+\alpha_m)$$
$$R(p) = (p+\beta_1)(p+\beta_2)\cdots(p+\beta_n)$$

and k is of no consequence. One or more of the α's, but none of the β's may be zero. Since the existence of the integral in (9) imposes the requirement that $F(t)$ have no discontinuities of higher type than finite jumps in the range $0- < t < \infty$, the continuity conditions on $W(t)$ in (10) must depend upon the difference between m and n in the expressions for $Q(p)$ and $R(p)$.

If $m \geq n$, it is fairly obvious that $W(t)$ must be differentiable, in the ordinary sense, exactly $m - n$ times. In other words, $W(t)$ and all its derivatives up to and including the $(m - n - 1)$th must be continuous, but the $(m - n)$th derivative may have finite jumps. If $m < n$ we must consider the introduction into $W(t)$ of discontinuities of higher type than finite jumps. These discontinuities arise in the formal extension of the concept of differentiation to functions containing finite jumps.

If a function $\phi(t)$ has a finite jump of amplitude A_0 at $t = a$, the value of $\phi'(t)$ at that point will be indicated formally as $A_0\cdot\delta_0(t-a)$ where $\delta_0(t-a)$ is a unit impulse at $t = a$. If $\phi'(a+0) - \phi'(a-0) = A_1$, the value of $\phi''(t)$ at $t = a$ will be indicated formally as $A_0\cdot\delta_1(t-a) + A_1\cdot\delta_0(t-a)$ where $\delta_1(t-a)$ is a unit doublet at $t = a$. And so on, for higher derivatives of $\phi(t)$.

The expression (9) is a minimum under the restriction (11) if $W(t)$ satisfies the differential equation

$$Q(p) \cdot Q(-p) \cdot W(t) = \text{const.} \qquad (12)$$

when $0 < t < 1$ and $Y(p)$ the condition

$$\frac{1}{2\pi i}\int_{i\infty}^{i\infty} S(p) \cdot S(-p) \cdot Y(p)e^{pt}dp = \text{const.}$$

when $0 < t < 1$. (13)

The restriction (11) itself requires that $W(t) = 0$ when $t > 1$, and

$$\int_{0-}^{1+} W(t)dt = 1. \qquad (14)$$

CASE I. $(n = 0)$

The general solution of (12) contains $2m+1$ constants of integration which are determined by (14) and the $2m$ continuity conditions that $W(t)$ and all of its derivatives up to and including the $(m-1)$th must vanish at $t = 0$ and $t = 1$.

CASE II. $(n \neq 0, m \geq n)$

The general solution of (12) contains $2m + 1$ constants of integration which are reduced to $2n$ in number by (14) and the $2(m - n)$ continuity conditions that $W(t)$ and all of its derivatives up to and including the $(m - n - 1)$th must vanish at $t = 0$ and at $t = 1$. The remaining $2n$ constants are determined by (13).

The left-hand member of (13) may be formulated by the method of residues. The expression for $Y(p)$ should first be separated into two parts so that

$$Y(p) = Y_L(p) + Y_R(p)e^{-p}$$

where $Y_L(p)$ and $Y_R(p)$ are rational functions of $S(p) \cdot S(-p) \cdot Y_L(p)e^{pt}$ in the left-hand in the left-hand half of the p-plane for the first part of $Y(p)$, and in the right-hand half for the second part. Hence, if the sum of the residues of $S(p) \cdot S(-p) \cdot Y_L(p)e^{pt}$ in the left-hand half of the p-plane be donated by $\Sigma_{L'}$ and if the sum of the residues of $S(p) \cdot S(-p) \cdot Y_R(p) \cdot e^{p(t-1)}$ in the right-hand half of the p-plane be denoted by $\Sigma_{R'}$ then the condition (13) reduces to

$$\Sigma_L - \Sigma_R = \text{const.} \qquad (15)$$

CASE III. $(n \neq 0, m < n)$

The $2m + 1$ constants of integration in the general solution of (12) are first increased to $2n + 1$ by appending the $2(n - m)$ singularities

$$\delta_0(t), \quad \delta_1(t), \quad \cdots \delta_{n-m-1}(t)$$

$$\delta_0(t-1), \delta_1(t-1), \cdots \delta_{n-m-1}(t-1)$$

and then reduced to $2n$ by (14). The remainder are determined by (13) or (15).

In formulating

$$Y(p) = L[W(t)]$$

it may be noted that

$$L[\delta_n(t-a)] = p^n e^{-ap} \qquad (a \geq 0) .$$

EXAMPLE OF CASE I

Let $S(p) = p^m$. The differential equation (12) requires $W(t)$ to be a polynomial of degree $2m$. The conditions at $t = 0$ require it to have a factor t^m, and those at $t = 1$, a factor $(1 - t)^m$. This leaves only (14) to be satisfied. Hence

$$W(t) = \frac{(2m+1)!}{(m!)^2}[t(1-t)]^m \qquad (0 \leq t \leq 1)$$

in agreement with (8) of Section 10.3.

EXAMPLE OF CASE II

Let
$$S(p) = \frac{p+\alpha}{p+\beta}. \qquad (12)$$

Then, by

$$W(t) = A_0 + A_1e^{-\alpha t} + A_2e^{\alpha t} \qquad (0 \leq t \leq 1).$$

Hence

$$Y(p) = \frac{A_0}{p} + \frac{A_1}{p+\alpha} + \frac{A_2}{p-\alpha} - \left[\frac{A_0}{p} + \frac{A_1e^{-\alpha}}{p+\alpha} + \frac{A_2e^{\alpha}}{p-\alpha}\right]e^{-p}$$

$$\Sigma_L = \frac{A_0\alpha^2}{\beta^2} - \left[A_0\frac{\alpha^2-\beta^2}{2\beta^2} - A_1\frac{\alpha+\beta}{2\beta} + A_2\frac{\alpha-\beta}{2\beta}\right]e^{-\beta t}$$

$$\Sigma_R = \left[A_0\frac{\alpha^2-\beta^2}{2\beta^2} + A_1\frac{\alpha-\beta}{2\beta}e^{-\alpha} - A_2\frac{\alpha+\beta}{2\beta}e_\alpha\right]e^{\beta(t-1)}.$$

Condition (15) is satisfied if

$$A_1 = \frac{1}{2}A_0\,Q\,e^{\alpha/2} \quad A_2 = \frac{1}{2}A_0\,Q\,e^{-\alpha/2}$$

where

$$Q = \frac{\alpha_2 - \beta_2}{\beta\left(\alpha \sinh \frac{\alpha}{2} + \beta \cosh \right)\frac{\alpha}{2}}.$$

Hence

$$W(t) = \frac{1 + Q \cosh \alpha\left(t - \frac{1}{2}\right)}{1 + \frac{2Q}{\alpha} \sinh \frac{\alpha}{2}} \qquad (0 \leq t \leq 1).$$

In the limit as $\alpha \to 0$, $S(p) = \dfrac{p}{p + \beta}$

and

$$W(t) = \frac{1 + \frac{\beta^2}{2 + \beta} t\,(1 - t)}{1 + \frac{1}{6} \frac{\beta^2}{2 + \beta}} \qquad (0 \leq t \leq 1).$$

In terms of expressions (12), Section 11.3.

$$W(t) = \frac{W_0(t) + kw_1(t)}{1 + k} \qquad (0 \leq t \leq 1)$$

where $k = 1/6[\beta^2/(2 + \beta)]$. This is reminiscent of Stibitz's results mentioned in Section 10.3.

EXAMPLE OF CASE III

Let $S(p) = 1/1 + \beta$. Then, by (12) and the rule for appending singularities in Case III

$$W(t) = A_0 + A_1\delta_0(t) + A_2\delta_0(t - 1) \qquad (0 \leq t \leq 1).$$

Hence

$$Y(p) = \frac{A_0 + A_1 p}{p} - \frac{A_0 - A_2 p}{p} e^{-p}$$

$$\Sigma_L = -\frac{A_0}{\beta^2} + \frac{A_0 - \beta A_1}{2\beta^2} e^{-\beta t}$$

$$\Sigma_R = -\frac{A_0 - \beta A_2}{2\beta_2} e^{\beta(t-1)}.$$

Condition (15) is satisfied if

$$A_1 = A_2 = \frac{A_0}{\beta}.$$

Hence

$$W(t) = \frac{1 + \frac{\delta_0(t) + \delta_0(t - 1)}{\beta}}{1 + \frac{2}{\beta}} \qquad (0 \leq t \leq 1).$$

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OEMsr–991 Supplement 7	Jam Handy Organization, Inc. Detroit, Michigan	Vector gunsight and assessing camera.	7.2–67
OEMsr–992 Supplement 5	General Electric Company Schenectady, New York	Airborne gunnery computers.	7.2–57
OEMsr–1008	Bausch & Lomb Optical Company Rochester, New York	Tank fire control.	7.4–66
OEMsr–1016 Supplement 1	Bausch & Lomb Optical Company Rochester, New York		
OEMsr–1044 Supplement 8	Librascope, Inc. Burbank, California	Development of computer for gun director Mark 56.	7.6–85
OEMsr–1059 Supplement 3	Brown University Providence, Rhode Island	Reticle design.	7.4–44
OEMsr–1116 Supplement 1	Keuffel & Esser New York, New York		
OEMsr–1137 Supplement 3	Bryant Chucking Grinder Company Springfield, Vermont	Mechanical director for 90 mm AA guns.	7.1–68
OEMsr–1144 Supplement 3	Foxboro Company Foxboro, Massachusetts	Development of steering mechanisms for torpedoes.	7.3–69
OEMsr–1160	Western Electric Company New York, New York	Relay interpolator.	7.5–70
OEMsr–1181	General Electric Company Schenectady, New York	Gyro unit for gun director Mark 56.	7.6–71

Contract Number	Contractor	Subject	Division Project Number
OEMsr-1185 Supplement 3	Waugh Equipment Company New York, New York	Mechanism to measure the smoothness of control of aircraft turrets.	7.3-75
OEMsr-1190 Supplement 6	Baker Manufacturing Company Evansville, Wisconsin	Course-invariant sights.	7.1-73
OEMsr-1208 Supplement 3	General Electric Company Schenectady, New York	Sea-borne torpedo director.	7.6-72
OEMsr-1235	Massachusetts Institute of Technology Servomechanisms Laboratory Cambridge, Massachusetts	Redesign of gun director Mark 49.	7.6-77
OEMsr-1236	Western Electric Company New York, New York	Automatic relay ballistic computers.	7.5-74
OEMsr-1237 Supplement 3	Columbia University New York, New York	Fire-control electronics.	7.2-76
OEMsr-1263 Supplement 5	Western Electric Company New York, New York	Development of computer T17.	7.1-78
OEMsr-1276 Supplement 4	Northwestern University Chicago, Illinois	Aircraft fire control analysis.	7.2-80
OEMsr-1292 Supplement 2	Leeds & Northrup Company Philadelphia, Pennsylvania	Speed regulator for motors and motor generators.	7.3-81
OEMsr-1299	General Electric Company Schenectady, New York	Development of gun director Mark 56.	7.6-79
OEMsr-1366 Supplement 4	Lawrance Aeronautical Corporation Linden, New Jersey	Control elements for fire control applications.	7.3-82
OEMsr-1387 Supplement 2	The Bristol Company Waterbury, Connecticut	Components for pilot operated sight.	7.2-84
OEMsr-1405	Westinghouse Electric Corporation Baltimore, Maryland	Study of employment of radio doppler effect for velocity determination of projectiles.	7.6-83
Symbol Numbers 1891, 1938	International Business Machine Corporation New York, New York	Torpedo director.	7.2-53

SERVICE PROJECT NUMBERS

The projects listed below were transmitted to the Executive Secretary, NDRC, from the War or Navy Department through either the War Department Liaison Officer for NDRC or the Office of Research and Inventions (formerly the Coordinator of Research and Development), Navy Department.

Service Project Number	*Subject*
AN-4	Angular velocity bombsight for use at low altitudes.
AN-5	Airborne fire control committee.
AC-27	Bombardier's calculator.
AC-28	Hydraulic controls for small caliber guns.
AC-47	Development of equipment for testing fire control apparatus.
AC-61	Development and construction of an own-speed computer.
AC-116	Determination of the most suitable method of testing flexible aerial gunsights.
AC-119	Procurement of one set of film assessment equipment.
AC-121	Airborne rocket sights.
AC-128	Airborne computer calibration.
OD-20	Development of a short-base wide-angle range finder suitable for use in moving vehicles.
OD-28	Electrical computing director for the 90 mm AA gun.
OD-41	Transmission of data from base end stations.
OD-51	Self collimating height finder.
OD-53	Reduction in errors in remote control system M1.
OD-55	"One-plus" type BTL electric antiaircraft director.
OD-56	Director for use with rocket projectors.
OD-67	Field artillery antitank fire control equipment.
OD-78	Investigation of Bush differential analyzer.
OD-89	Use of nonreflecting films on height finder optics.
OD-91	Modification of director M7.
OD-93	Computing sight T20.
OD-100	Study of employment of radio doppler effect for velocity determination of projectiles.
OD-101	Development of a new type stereoscopic range finder.
OD-104	Development of computing sight T31.
OD-105	Test data recorder.
OD-111	Triangle solver.
OD-120	Intermediate range director M5A1E1.
OD-122	Smoothing device for employment with SCR-268.
OD-127	Extend the utility of the Barber-Colman dynamic tester.
OD-132	Plotting board T9.
OD-142	Intermediate caliber antiaircraft range director T28.

Service Project Number	*Subject*
OD-153	Automatic relay ballistic computers.
OD-157	Development of computer T17.
OD-203	Blind firing installation for 40 mm gun using director and radar.
D2-40	Gyroscopic director development.
NA-114	Construction of two machines for testing aircraft fire control equipment.
NA-136	Adaptation of an existing electronic remote control system to a Bell Sunstrand hydraulic turret drive unit.
NA-158	Mechanism to measure the smoothness of control of aircraft turrets.
NA-161	Aircraft fire control analyses—Naval Air Station, Patuxent River.
NA-168	Slant range computer.
NA-203	Design of computing mechanism for the torpedo deflection trainer.
NA-232	Development of Razon attachment for Navy 7-A-3 Trainer.
NO-106	Aircraft torpedo director.
NO-107	Electronic computer.
NO-108	Probability and statistical study of plane-to-plane fire.
NO-109	Methods for operating equipment related to fire control apparatus.
NO-112	Short-base antiaircraft range finders.
NO-113	Directing tracer fire in plane-to-plane firing.
NO-122	Development of a gyro lead-computing tail turret gunsight.
NO-128	Projected plane gunsight.
NO-129	Antisubmarine bombsight.
NO-130	(No title available.)
NO-131	Probability studies.
NO-134	Mark 32 (TBF type) torpedo director for motor torpedo boats.
NO-136	(No title available.)
NO-137	Comparative study, three types of reticles.
NO-139	Stereoscopic range finder.
NO-145	(No title available.)
NO-152	Development of gunsight computer for flexible aerial gunnery.
NO-154	(No title available.)
NO-166	Development of gun director Mk 56 (Division 14).
NO-178	Tracking tests on optical ring sights.
NO-179	Development of steering mechanisms for torpedoes.
NO-180	Maneuverable target for Mark 2 bombing trainer.
NO-186	To provide target course stabilization for the torpedo director Mk 30.
NO-190	Airship bombsight.
NO-191	A preset attachment for the Mark 15 bombsight.
NO-197	Design of fire control for torpedoes for destroyers and light cruisers.
NO-203	Redesign of gun director Mark 49.

Service Project Number	*Subject*
NO-207	Improvements to the gun director Mark 37.
NO-209	Stabilized roll indicator (with Section 6.1).
NO-216	Adaptations of gunsight Mk 18 for other purposes.
NO-240	NDRC advisory service on radar bombsight.
NO-241	Small combination aircraft torpedo director and low altitude bombsight.
NO-242	Range-type torpedo director with provision for servo range input.
NO-243	Torpedo bombsight for high altitude torpedo release.
NO-244	Mechanization of a correction for maneuvering targets.
NO-255	Illuminated sight Mk 14.
NO-265	Multiple purpose pilot's sighting system.
NO-268	Study of aircraft turrets equipped with lead computing sights and aided tracking.
NS-334	Advisory services of Division 7 for Bureau of Ships.

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